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Open Wilson lines and group theory of noncommutative Yang-Mills theory in two dimensions

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Abstract: The correlation functions of open Wilson line operators in two-dimensional Yang-Mills theory on the noncommutative torus are computed exactly. The correlators are expressed in two equivalent forms. An instanton expansion involves only topological numbers of Heisenberg modules and enables extraction of the weak-coupling limit of the gauge theory. A dual algebraic expansion involves only group theoretic quantities, winding numbers and translational zero modes, and enables analysis of the strong-coupling limit of the gauge theory and the high-momentum behaviour of open Wilson lines. The dual expressions can be interpreted physically as exact sums over contributions from virtual electric dipole quanta.

Keywords: Nonperturbative Effects, Strong Coupling Expansion, Non-Commutative Geometry.

Contents

1. Introduction and summary

Field theories on noncommutative spaces continue to challenge the conventional wisdom of quantum field theory (see [1]–[3] for reviews and exhaustive lists of references). They can be naturally embedded into string theory and as such possess many unusual non-local effects which are not observed in their commutative counterparts. Striking differences between these models and ordinary quantum field theories are readily seen in perturbation theory. The perturbation series exhibits poles in the noncommutativity parameter θ at each finite order, and mixes ultraviolet and infrared modes in a way that appears to make conventional renormalization schemes inapplicable to this class of field theories. However, such non-analyticities and instabilities may simply be an artifact of the perturbative approximation. In order to truly understand how noncommutative field theories differ from ordinary ones, and what they can teach us about string dynamics in background fields, it is important to have a good understanding of the non-perturbative physics of these models.

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A good place to look for exactly solvable models is in two spacetime dimensions. In particular, inspired by the wealth of methods that can be used to solve ordinary gauge theory in two dimensions (see [4] for a review), the gauge invariant correlation functions of two-dimensional noncommutative Yang-Mills theory have been studied from various different points of view with an eye towards understanding generic features of its exact solution. Perturbative studies of the analytic structure of Wilson loop correlators may be found in [5]–[7]. Fluxon representations in the large-N limit of open Wilson line correlators are given in [8]. The high-energy behaviour of open Wilson lines at large N is studied within the ladder approximation in [9]. Numerical simulations of both Wilson loops and open Wilson lines based on the reduced model representation of noncommutative gauge theory may be found in [10]–[12].

In this paper we shall build on the exact non-perturbative expression for the vacuum amplitude of two-dimensional gauge theory on the noncommutative torus that was obtained in [13]. We shall present two main achievements. First of all, we will derive the exact expression for the pair correlator of open Wilson line operators for the first time. This will exhibit the strength of the technique proposed in [13] at obtaining analytic results in gauge theory on the noncommutative torus. Secondly, we will derive an expansion dual to that of [13] which is analytic in the Yang-Mills coupling constant. This series elucidates precisely how noncommutativity alters gauge theory in two dimensions. In the commutative case, the partition function and observables can be evaluated by using heat kernel methods, leading to an explicit expansion into irreducible representations of the gauge group which may be parameterized by integer-valued Young tableaux row variables. In the noncommutative case, the resummed partition function shows that the group theory row integers are promoted to doublets of integers which generically cannot be decoupled. The open Wilson line calculations then serve to show that the promoted group theory doublet can be interpreted as consisting of a representation (winding) integer plus a translational zero mode (momentum). This gives a very precise demonstration of how noncommutativity interlaces colour and spacetime degrees of freedom. These calculations also provide a basis for potentially understanding how to make sense of the large-N limit of the solution to gauge theory on the noncommutative torus, and hence a possible realization of a string representation of two-dimensional noncommutative Yang-Mills theory. Whether or not a sensible and non-trivial large-N limit is possible on the noncommutative torus is presently not clear, and this issue deserves further investigation.

1.1 Outline and summary of results

In the remainder of this section we will sketch the structure of the rest of the paper and summarize our main findings. All facts used in this paper concerning noncommutative gauge theory can be found in [13] and the review articles [1]–[3]. In section 2 we will start from the expansion, derived in [13], of the partition function of noncommutative Yang-Mills theory in terms of partitions of the topological numbers characterizing a projective module over the noncommutative torus. We will re-interpret this series in terms of a dual description which involves the integer-valued charges and momenta of collections of virtual electric dipoles, the non-local quanta that characterize generic noncommutative field theo-

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ries [14, 15]. One advantage of this dual representation is that with it the partition function is manifestly an analytic function of the noncommutativity parameter, thus showing explicitly that it is smooth in θ. The simplifications which occur in this expansion for rational values of the noncommutativity parameter are investigated and the $\mathrm{SL}(2,\mathbb{Z})$ transformation properties of the dual electric and momentum charges are derived. For irrational values of the noncommutativity parameter such simplifications are not possible, although a rational limiting process produces an expression for the partition function which is reminiscent of a large-N matrix model. Finally, we investigate an expansion in small values of θ and show that noncommutative Yang-Mills theory is asymptotically equivalent to generalized Yang-Mills theory [16]–[18] with exponentially small corrections. It is important here that the series is only asymptotic as expected from renormalizability arguments.

In section 3 we turn our attention to the study of non-local observables in ordinary Yang-Mills theory on the torus. In particular, we consider the pair correlator of simple Polyakov loops with integer winding number. Using some well-known machinery, we derive an exact expression for these correlation functions in terms of an instanton expansion. This represents the first calculation of the Polyakov loop anti-loop correlator in the instanton representation of commutative Yang-Mills theory. The Fourier dual of this result produces a momentum-dependent loop correlator which involves structures that are native to noncommutative geometry.

In section 4 we analyse the pair correlators of open Wilson lines in the noncommutative case, which yield the interaction amplitudes between pairs of electric dipoles. The correlation functions are computed, via the arguments of [13], exactly for all values of the noncommutativity parameter θ by generalizing the momentum space Polyakov loop pair correlator to noncommutative space using Morita covariance under $\mathrm{SL}(2,\mathbb{Z})$ transformations of the topological and dual charge-momentum integers. We illustrate how the resulting correlation functions exhibit structures which appear in BPS configurations of dipoles.

With an exact expression for the pair correlator at hand, in section 5 we proceed to investigate its properties at both weak and strong coupling. The weak-coupling limit of the correlator is shown to depend on an $\mathrm{SL}(2,\mathbb{Z})$-invariant pairing of the topological and charge-momentum integers in K-theory. Physically, the weak-coupling results reveal new insights into commutative Yang-Mills theory with correlation functions presented in a momentum representation. In particular, it is shown that even the commutative theory naturally depends on both charge and momentum quantum numbers which characterize BPS states. We also compute the strong-coupling expansion of the correlation function of open Wilson lines. This form gives the clearest perspective on the dual nature of the topological and charge-momentum integers, and it makes explicit the twisting together of colour-electric and spacetime degrees of freedom in the noncommutative gauge theory. Moreover, it can be used to show that the correlator is a smooth function of θ, thereby suggesting that any non-analyticities in perturbative calculations are resummable artifacts.

Finally, in section 6 we investigate the high-momentum dependence of the correlators. We show that the correlation functions of open Wilson line operators in two-dimensional noncommutative Yang-Mills theory are uniformly bounded in the Yang-Mills coupling con-

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stant. In particular, the correlator does not exhibit any exponential growth with momentum like its four-dimensional counterpart does [19]–[21]. In fact, we establish explicitly that the correlation functions vanish in the limit of large momentum.

2. The instanton expansion and group theoretic deformations

The partition function of two-dimensional quantum gauge theory on the noncommutative torus is given by the euclidean functional integral

$$Z_{p,q}(A,\theta) = \int \mathrm{D}\mathcal{A}\ \exp\left[-\frac{1}{2g^2}\int \mathrm{d}^2x\ \mathrm{Tr}\left(F-\frac{2\pi^2\,b_{p,q}}{A}\right)^2\right], \tag{2.1}$$

where g is the Yang-Mills coupling constant and A is the area of the square torus $\mathbf{T}^2$. Here $F = \partial_1\mathcal{A}_2 - \partial_2\mathcal{A}_1 + \mathcal{A}_1 \star \mathcal{A}_2 - \mathcal{A}_2 \star \mathcal{A}_1$ is the noncommutative field strength tensor which is defined using the usual star-product with a dimensionless noncommutativity parameter θ. The anti-hermitean $\mathrm{U}(N)$ gauge field $\mathcal{A}$ defines a connection of a fixed Heisenberg module over the noncommutative torus of topological numbers $(p,q)\in\mathbb{Z}^2$, dimension $p-q\,\theta>0$, and rank $N=\gcd(p,q)$. In the commutative case $\theta=0$, this makes a distinction between physical Yang-Mills theory, which takes into account the sum over all magnetic fluxes q (i.e. all topological classes of principal $\mathrm{U}(p)$ bundles over $\mathbf{T}^2$), and Yang-Mills theory defined on a fixed projective module. In the noncommutative case, only the latter gauge theory appears to be amenable to an unambiguous definition, as the topological numbers (p,q) do not generically decouple in this case. We have also subtracted the constant curvature

$$b_{p,q} = \frac{q}{p-q\,\theta} \tag{2.2}$$

of the module in (2.1) which corresponds to the global minimum of the noncommutative Yang-Mills action. It is a topological term whose only essential role is to ensure that the action is invariant under gauge Morita duality transformations.

The functional integral (2.1) is given exactly by its semi-classical approximation, owing to a hidden supersymmetry present in the path integral which kills higher quantum loop corrections. It can thereby be represented as a sum over solutions of the classical noncommutative Yang-Mills equations. This leads to the explicit expansion [13]

$$\begin{aligned} Z_{p,q}(A,\theta) = &\sum_{\substack{\text{partitions}\\(\boldsymbol{p},\boldsymbol{q})}} \frac{(-1)^{|\boldsymbol{\nu}|}}{\prod_a \nu_a!}\prod_{k=1}^{|\boldsymbol{\nu}|}\sqrt{\frac{2\pi^2}{g^2A\,(p_k-q_k\,\theta)^3}}\ \times \\ &\times \exp\left[-\frac{2\pi^2}{g^2A}\sum_{k=1}^{|\boldsymbol{\nu}|}(p_k-q_k\,\theta)\left(\frac{q_k}{p_k-q_k\,\theta}-\frac{q}{p-q\,\theta}\right)^2\right], \end{aligned} \tag{2.3}$$

where the sum is over all "partitions" of the topological numbers (p,q), which are the collections of integers $(\boldsymbol{p},\boldsymbol{q})=\{(p_k,q_k)\}$ obeying the constraints

$$p_k - q_k\,\theta > 0\,,$$

$$
\sum_k \left(p_k - q_k\,\theta\right) = p - q\,\theta\,,
$$
$$
\sum_k q_k = q\,. \tag{2.4}
$$

The integer $\nu_a > 0$ is the number of partition components $(p_k, q_k) \in (\boldsymbol{p}, \boldsymbol{q})$ which have the $a^{\rm th}$ least dimension $p_a - q_a\,\theta$, while $|\boldsymbol{\nu}| = \nu_1 + \nu_2 + \cdots$ is the total number of components in the partition $(\boldsymbol{p}, \boldsymbol{q})$.[1] When $\theta = 0$, the last constraint in (2.4) distinguishes the partition function (2.3) from that of physical Yang-Mills theory. Physical commutative Yang-Mills theory is realized as an appropriately weighted sum over sectors with distinct Chern numbers q.

The set of all partitions for a given Heisenberg module are in one-to-one correspondence with classical gauge field configurations. The exponential factor in (2.3) is the classical contribution to the Yang-Mills partition function (2.1). By resumming (2.3) over gauge inequivalent partitions [13], we may thereby express it as a sum over unstable instantons. The exponential prefactors in (2.3) then represent the *exact* corrections due to quantum fluctuations around a classical solution and are determined by a finite, non-trivial perturbative expansion about each instanton.

2.1 The dual expansion

The expansion (2.3) is inherently non-perturbative as it contains terms of order $\mathrm{e}^{\,-1/g^2A}$. It is natural to ask if there exists a corresponding "dual" expansion which is analytic in the Yang-Mills coupling constant. For this, we will drop the background flux $b_{p,q}$ in (2.1) for simplicity and rewrite the partition sum (2.3) as series over *unconstrained* integers as

$$
\begin{aligned}
Z_{p,q}(A,\theta) = {} & \sum_{\boldsymbol{p},\boldsymbol{q}} \frac{(-1)^{|\boldsymbol{\nu}|}}{\prod_a \nu_a!}\;\delta_{p\,,\sum_k p_k}\;\delta_{q\,,\sum_k q_k}\,\times \\
& \times \prod_{k=1}^{|\boldsymbol{\nu}|}\sqrt{\frac{2\pi^2}{g^2A}}\;\int_{0+}^{\infty}\frac{\mathrm{d}z_k}{(z_k)^{3/2}}\;\delta(z_k - p_k + q_k\,\theta)\;\mathrm{e}^{\,-\frac{2\pi^2}{g^2A}\,\frac{(q_k)^2}{z_k}}\,.
\end{aligned} \tag{2.5}
$$

The integrations over $z_k > 0$ enforce the dimension positivity constraints in (2.4), while the Kronecker delta-functions enforce the last two constraints. By resolving the delta-functions over the topological charges q_k and the dimension variables z_k we get

$$
\begin{aligned}
Z_{p,q}(A,\theta) = {} & \sum_{\boldsymbol{p},\boldsymbol{q}} \frac{(-1)^{|\boldsymbol{\nu}|}}{\prod_a \nu_a!}\delta_{p\,,\sum_k p_k}\int_0^1 \mathrm{d}\lambda\;\mathrm{e}^{\,-2\pi\,\mathrm{i}\,\lambda\,q}\,\times \\
& \times \prod_{k=1}^{|\boldsymbol{\nu}|}\sqrt{\frac{2\pi^2}{g^2A}}\;\int_{0+}^{\infty}\frac{\mathrm{d}z_k}{(z_k)^{3/2}}\int_{-\infty}^{\infty}\frac{\mathrm{d}x_k}{2\pi}\;\mathrm{e}^{\,\mathrm{i}\,x_k(z_k - p_k + q_k\,\theta)}\;\mathrm{e}^{\,2\pi\,\mathrm{i}\,\lambda\,q_k - \frac{2\pi^2}{g^2A}\,\frac{(q_k)^2}{z_k}}\,.
\end{aligned} \tag{2.6}
$$

[1]Notation: For a given partition, the integer $k = 1, \ldots, |\boldsymbol{\nu}|$ labels its components (p_k, q_k), while a labels the degeneracy integers ν_a. Since the coupling constant and area of the torus enter the partition function only through the dimensionless combination g^2A, for brevity we display explicitly only the area dependence of all observables.

The q_k series are gaussian sums, and so they can each be rewritten by using the Poisson resummation formula

$$\sum_{m=-\infty}^{\infty} \mathrm{e}^{\,-\pi h m^2 - 2\pi\,\mathrm{i}\,b\,m} = \frac{1}{\sqrt{h}}\,\sum_{q=-\infty}^{\infty} \mathrm{e}^{\,-\pi (q-b)^2/h}\,. \tag{2.7}$$

Then the resulting integrations over x_k are also gaussian, and we find

$$\begin{aligned} Z_{p,q}(A,\theta) \,&=\, \sum_{\boldsymbol{p},\boldsymbol{m}} \frac{(-1)^{|\boldsymbol{\nu}|}}{\prod_a \nu_a!}\,\delta_{p\,,\,\sum_k p_k}\,\int_0^1 \mathrm{d}\lambda\,\mathrm{e}^{\,-2\pi\,\mathrm{i}\,\lambda\,q}\,\times \\ &\times\prod_{k=1}^{|\boldsymbol{\nu}|}\sqrt{\frac{2\pi^2}{g^2 A\,\theta^2}}\int_{0+}^{\infty}\frac{\mathrm{d}z_k}{(z_k)^{3/2}}\,\mathrm{e}^{\,-\frac{g^2 A}{2}\,z_k(m_k+\lambda)^2}\,\times \\ &\times\exp\left[-\frac{2\pi^2}{g^2 A\,\theta^2\,z_k}\,\left(p_k - z_k - \mathrm{i}\,\frac{g^2 A\,\theta\,z_k}{2\pi}\,\Big(\lambda+m_k\Big)\right)^2\right]. \end{aligned} \tag{2.8}$$

Now we repeat the same procedure for the sum over the p_k's in (2.8). Fourier resolving the constraints on the p_k's by a circular coordinate $\mu \in [0,1]$ leaves a gaussian sum in each p_k, which may be Poisson resummed using (2.7) in terms of dual integers n_k. This gives

$$\begin{aligned} Z_{p,q}(A,\theta) \,&=\, \sum_{\boldsymbol{n},\boldsymbol{m}} \frac{(-1)^{|\boldsymbol{\nu}|}}{\prod_a \nu_a!}\int_0^1 \mathrm{d}\mu\;\mathrm{e}^{\,-2\pi\,\mathrm{i}\,\mu\,p}\int_0^1 \mathrm{d}\lambda\;\;\mathrm{e}^{\,-2\pi\,\mathrm{i}\,\lambda\,q}\,\times \\ &\times\prod_{k=1}^{|\boldsymbol{\nu}|}\int_{0+}^{\infty}\frac{\mathrm{d}z_k}{z_k}\;\;\mathrm{e}^{\,-\frac{g^2 A}{2}\,z_k\left(m_k+\lambda+\theta\,(\mu-n_k)\right)^2}\;\;\mathrm{e}^{\,2\pi\,\mathrm{i}\,z_k(\mu-n_k)}\,. \end{aligned} \tag{2.9}$$

We now shift the circular coordinate $\lambda \to \lambda - \theta\,\mu$ in order to eliminate the μ-dependence of the quadratic exponent in (2.9). Then the shifted integration range is $\lambda \in [\theta\,\mu, 1+\theta\mu]$. But the integrand of (2.9) is easily seen to be a periodic function of λ with period 1. As a consequence, since the λ integral goes over a single period, it is independent of the offset $\theta\,\mu$. In this way we arrive at our final expression for the resummed, dual partition function as

$$\begin{aligned} Z_{p,q}(A,\theta) \,&=\, \sum_{\boldsymbol{n},\boldsymbol{m}} \frac{(-1)^{|\boldsymbol{\nu}|}}{\prod_a \nu_a!}\int_0^1 \mathrm{d}\mu\;\;\mathrm{e}^{\,-2\pi\,\mathrm{i}\,\mu\,(p-q\,\theta)}\int_0^1 \mathrm{d}\lambda\,\mathrm{e}^{\,-2\pi\,\mathrm{i}\,\lambda\,q}\,\times \\ &\times\prod_{k=1}^{|\boldsymbol{\nu}|}\int_{0+}^{\infty}\frac{\mathrm{d}z_k}{z_k}\;\;\mathrm{e}^{\,-\frac{g^2 A}{2}\,z_k(m_k-n_k\,\theta+\lambda)^2}\;\;\mathrm{e}^{\,2\pi\,\mathrm{i}\,z_k(\mu-n_k)}\,. \end{aligned} \tag{2.10}$$

Note that in this expansion the multiplicities ν_a are associated with the distinct numbers $m_a - n_a\,\theta$.

This dual form makes manifest the partition function as an analytic function of the noncommutativity parameter θ, as was proven in [13] directly from the original instanton expansion (2.3). Furthermore, while the instanton form (2.3) is useful for extracting the weak coupling limit of the gauge theory [13], the resummed version (2.10) makes tractable the large area (strong coupling) limit, from which we may conclude that the partition function on the noncommutative plane is a constant independent of all coupling parameters. As we will see below, it also allows one to make some explicit substantial simplifications of $Z_{p,q}(A,\theta)$ for rational values[2] of θ. In this way the dual expansion (2.10) will give a very precise description of how rational noncommutative gauge theories relate to commutative ones.

We will see later on that the quantities $m_k - n_k\theta$ and n_k in (2.10) correspond to electric flux and momentum, respectively, carried by the elementary quanta of the noncommutative gauge theory. The dual expansion thereby rewrites the vacuum energy in terms of contributions from virtual electric dipoles, with the quadratic terms in the exponential representing their zero-mode kinetic energy. The Poisson duality used above may therefore be regarded as an electric-magnetic duality which exchanges instantons and dipoles. In this section we shall focus on the group theoretical aspects of the strong-coupling expansion (2.10).

2.2 Group theory at $\theta = 0$

To get a feel for what the dual instanton expansion (2.10) represents, let us briefly consider the commutative case $\theta = 0$ and recall the well-known exact solution in this case in terms of group theoretical techniques [4]. We start from *ordinary* $\mathrm{U}(p)$ Yang-Mills theory on a finite cylinder within the hamiltonian formalism, whose Hilbert space of physical states is the space of L^2-class functions with respect to the normalized, invariant Haar measure $[\mathrm{d}U]$ on the unitary group $\mathrm{U}(p)$. A basis for this Hilbert space is provided by the characters

$$\chi_R(U) = \mathrm{Tr}_R\, U \tag{2.11}$$

in the irreducible unitary representations R of $\mathrm{U}(p)$, where U is the holonomy of the gauge field around the cycle of the cylinder at some fixed time slice. The hamiltonian is diagonalized in this representation basis and is essentially the laplacian on the group manifold of $\mathrm{U}(p)$. Its eigenvalue on the wavefunction (2.11) is proportional to the second Casimir invariant $C_2(R)$ of the representation R.

From these facts one can immediately write down the cylinder amplitude corresponding to propagation between two states characterized by holonomies U_1 and U_2 as

$$K_p(A;U_1,U_2) = \sum_R \chi_R(U_1)\, \chi_R^*(U_2)\, \mathrm{e}^{-\frac{g^2 A}{2}\, C_2(R)}\,, \tag{2.12}$$

where A is the area of the cylinder. This is just the standard heat kernel on the $\mathrm{U}(p)$ group. We can now glue the two ends of the cylinder together by setting $U_1 = U_2 = U$ and integrate over U using the fact that characters are orthonormal with respect to integration over the Haar measure of $\mathrm{U}(p)$,

$$\int_{\mathrm{U}(p)} [\mathrm{d}U]\, \chi_R(U)\, \chi_{R'}^*(U) = \delta_{R,R'}\ . \tag{2.13}$$

[2] We stress that such simplifications for rational values of θ do *not* indicate distinct behaviour of noncommutative gauge theory on the two-torus as compared to irrational values of θ. As demonstrated in [13], all derivatives with respect to θ of the partition function and simply related observables are finite.

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This gives the partition function for Yang-Mills theory on a torus $\mathbf{T}^2$ of area A as the vacuum amplitude

$$Z_p(A) = \int_{\mathrm{U}(p)} [\mathrm{d}U]\, K_p(A;U,U) = \sum_R \mathrm{e}^{\,-\frac{g^2 A}{2}\, C_2(R)}\,. \tag{2.14}$$

One can make the sum in (2.14) explicit by noting that the irreducible representations of the $\mathrm{U}(p)$ structure group are parameterized by decreasing sets $\boldsymbol{m} = (m_1, \dots, m_p)$ of p integers

$$+\infty > m_1 > m_2 > \cdots > m_p > -\infty \tag{2.15}$$

which are shifted highest weights that determine the lengths of the rows of the corresponding Young tableaux. Up to an irrelevant constant, the quadratic Casimir is given in terms of these row integers as

$$C_2(R) = C_2(\boldsymbol{m}) = \sum_{a=1}^{p} \left(m_a - \frac{p-1}{2}\right)^2 \tag{2.16}$$

and is symmetric under permutations of the m_a's. One can thereby write the sum in (2.14) as a sum over all non-coincident integers $m_1 \neq m_2 \neq \cdots \neq m_p$, which can then be turned into a sum over *all* $\boldsymbol{m} \in \mathbb{Z}^p$ by inserting

$$\det_{1\le a,b\le p} \left(\delta_{m_a,m_b}\right) = \sum_{\sigma\in S_p} (-1)^{|\sigma|} \prod_{a=1}^{p} \delta_{m_a,m_{\sigma(a)}}\,. \tag{2.17}$$

The permutation symmetry of (2.16) then implies that (2.17) truncates to a sum over conjugacy classes $[1^{\nu_1}\, 2^{\nu_2} \cdots p^{\nu_p}]$ of the symmetric group S_p which are labelled by partitions $\boldsymbol{\nu} = (\nu_1, \dots, \nu_p)$ of p, with $\nu_a \geq 0$ the number of elementary cycles of length a in $[1^{\nu_1}\, 2^{\nu_2} \cdots p^{\nu_p}]$ satisfying

$$\nu_1 + 2\,\nu_2 + \cdots + p\,\nu_p = p\,. \tag{2.18}$$

In this way, by dropping some irrelevant factors, (2.14) becomes

$$Z_p(A) = \sum_{\boldsymbol{m}\in\mathbb{Z}^p}\ \sum_{\substack{\text{partitions}\\ \boldsymbol{\nu}}}\ \prod_{a=1}^{p} \frac{(-1)^{\nu_a}}{a^{\nu_a}\,\nu_a!}\, \mathrm{e}^{\,-\frac{g^2 A}{2}\, a\, C_2\left(m_{\nu_1+\cdots+\nu_{a-1}+1}\,,\,\dots\,,\,m_{\nu_1+\cdots+\nu_a}\right)}\,, \tag{2.19}$$

where the exponential prefactor comes from the sign and order of the conjugacy class $[1^{\nu_1}\, 2^{\nu_2} \cdots p^{\nu_p}]$, and we have defined $\nu_0 \equiv 0$.

After repeated applications of the Poisson resummation formula (2.7), we arrive at the expansion (2.3) in the commutative case $\theta = 0$, with a weighted sum over all topological charges q,

$$Z_p(A) = \sum_{q=-\infty}^{\infty} \mathrm{e}^{\,\pi\,\mathrm{i}\,(p-1)\,q}\, Z_{p,q}(A,0)\,. \tag{2.20}$$

The details of this computation may be found in [13]. Note that at the level of the expansion (2.19), the definition of partition is slightly different from that given at the beginning of this section, because the magnetic charge q does not enter explicitly into the physical partition function. Namely, here the partitions are the collections of integers $\boldsymbol{p} = \{p_k\}$, $0 < p_k \leq p$ associated with the conjugacy classes of the symmetric group S_p, each component of $\boldsymbol{p}$ specifying the length of a cycle in a class. The first two constraints in (2.4) for $\theta = 0$ are still met, but now we allow for vanishing ν_a, the number of components in $\boldsymbol{p}$ of magnitude a.

Let us now return to (2.10) in light of this analysis. In the limit of vanishing noncommutativity parameter, the sums over m_k and n_k in (2.10) decouple. The series over n_k yields a periodic delta-function

$$\sum_{n_k=-\infty}^{\infty} \mathrm{e}^{\,2\pi\,\mathrm{i}\,z_k n_k} = \sum_{p_k=1}^{\infty} \delta(z_k - p_k)\,, \tag{2.21}$$

where we have used the fact that $z_k > 0$. Then the integration over μ in (2.10) imposes the constraint $\sum_k p_k = p$, i.e. that $\boldsymbol{p} \in \mathbb{Z}_+^p$ form a partition of the rank p of the structure group. The sign factors in (2.10) ensure that no pairs of integers m_k coincide, and the integers p_k enumerate coincident values of the m_k. By using the symmetry of (2.10) under arbitrary permutations of the m_k, summing over the p_k thereby produces the dual expansion of the commutative partition function as

$$Z_{p,q}(A,0) = \int_0^1 \mathrm{d}\lambda\,\mathrm{e}^{\,-2\pi\,\mathrm{i}\,\lambda\,q} \sum_{m_1>\cdots>m_p} \mathrm{e}^{\,-\frac{g^2A}{2}\,\sum_a (m_a+\lambda)^2}\,. \tag{2.22}$$

The weighted sum (2.20) of (2.22) reproduces the physical Yang-Mills partition function (2.14) as a sum over the Young tableaux weights (2.15) with Casimir invariant (2.16). Generally, (2.22) represents the heat kernel expansion of commutative Yang-Mills theory on a torus with the integration enforcing the constraints, within an algebraic setting, on the topological numbers of the given projective module.

2.3 Gauge Morita equivalence at rational θ

If the noncommutativity parameter θ is a rational number then it is possible to explicitly demonstrate a simplified presentation of the partition function (2.10). This does not imply the naive expectation that the behaviour of the theory fluctuates wildly as θ is tuned from rational to irrational values. It merely shows that, for certain special cases, gauge theory on the noncommutative two torus can be understood completely in terms of commutative gauge theory appropriately restricted to a topological sector.

For integer values $\theta = r \in \mathbb{Z}$ of the noncommutativity parameter, we can eliminate the n_k dependence of the quadratic exponent in (2.10) by shifting m_k by the integer $r\,n_k$, and in the same manner as in the previous subsection recover the heat kernel expansion of commutative $\mathrm{U}(p - rq)$ gauge theory. This shows explicitly that we may restrict attention to noncommutativity parameters in the range $\theta \in [0, 1)$, as the integer part of θ can always be shifted away.

For rational values $\theta = r/s$, with $r < s$ relatively prime positive integers, one can simplify (2.10) to a finite weighted sum over contributions from commutative Yang-Mills theory. For this, we consider separately the subseries $n_k = s\,d_k + a_k$, where $d_k \in \mathbb{Z}$ and $a_k = 0, 1, \ldots, s-1$. The partition function (2.10) is then given by the sum over d_k and a_k, and again by shifting m_k by the integer $r\,d_k$ we can decouple the sums over d_k to get

$$Z_{p,q}\Big(A, \frac{r}{s}\Big) = \sum_{\boldsymbol{d},\boldsymbol{m}} \sum_{a_1,a_2,\ldots=0}^{s-1} \frac{(-1)^{|\boldsymbol{\nu}|}}{\prod_a \nu_a!} \int_0^1 \mathrm{d}\mu\, \mathrm{e}^{\,-2\pi\,\mathrm{i}\,\mu(p-rq/s)} \int_0^1 \mathrm{d}\lambda\, \mathrm{e}^{\,-2\pi\,\mathrm{i}\,\lambda\,q} \times$$
$$\times \prod_{k=1}^{|\boldsymbol{\nu}|} \int_{0^+}^{\infty} \frac{\mathrm{d}z_k}{z_k}\, \mathrm{e}^{\,-\frac{g^2 A}{2} z_k (m_k - r a_k/s + \lambda)^2}\, \mathrm{e}^{\,2\pi\,\mathrm{i}\,z_k(\mu - s\,d_k - a_k)}\,. \tag{2.23}$$

The series over d_k and integral over z_k force $s\,z_k$ to be a positive integer p'_k, giving

$$Z_{p,q}\Big(A, \frac{r}{s}\Big) = \sum_{\boldsymbol{p}':p'_k>0} \sum_{\boldsymbol{m}} \sum_{a_1,a_2,\cdots=0}^{s-1} \frac{(-1)^{|\boldsymbol{\nu}|}}{\prod_a \nu_a!} \int_0^1 \mathrm{d}\mu\, \mathrm{e}^{\,-2\pi\,\mathrm{i}\,\mu(p-rq/s)} \int_0^1 \mathrm{d}\lambda\, \mathrm{e}^{\,-2\pi\,\mathrm{i}\,\lambda q} \times$$
$$\times \prod_{k=1}^{|\boldsymbol{\nu}|} \frac{s}{p'_k}\, \mathrm{e}^{\,-\frac{g^2 A}{2s}\, p'_k (m_k - r\,a_k/s + \lambda)^2}\, \mathrm{e}^{\,2\pi\,\mathrm{i}\,p'_k(\mu - a_k)/s}\,. \tag{2.24}$$

After rescaling $\mu \to s\,\mu$, the integrand of (2.24) is invariant under integer shifts of the circular coordinate μ. Averaging (2.24) over the integer shifts in μ from 0 to $s-1$ then yields a manifestly $\mathbb{Z}$-periodic expression in μ as

$$Z_{p,q}\Big(A, \frac{r}{s}\Big) = \sum_{\boldsymbol{p}':p'_k>0} \sum_{\boldsymbol{m}} \sum_{a_1,a_2,\cdots=0}^{s-1} \frac{(-1)^{|\boldsymbol{\nu}|}}{\prod_a \nu_a!} \int_0^1 \mathrm{d}\mu\, \mathrm{e}^{\,-2\pi\,\mathrm{i}\,\mu(sp-rq)} \int_0^1 \mathrm{d}\lambda\, \mathrm{e}^{\,-2\pi\,\mathrm{i}\,\lambda q} \times$$
$$\times \prod_{k=1}^{|\boldsymbol{\nu}|} \frac{s}{p'_k}\, \mathrm{e}^{\,-\frac{g^2 A}{2s}\, p'_k (m_k - r\,a_k/s + \lambda)^2}\, \mathrm{e}^{\,2\pi\,\mathrm{i}\,p'_k(\mu - a_k/s)}\,. \tag{2.25}$$

It is now clear from (2.25) that the integration over μ enforces the constraint

$$\sum_k p'_k = sp - rq \equiv p'\,. \tag{2.26}$$

As in the previous subsection, we will express this partitioning of the integer p', along with the sign factors in (2.25), with p'_k enumerating coincident values of the sets of integers m_k and a_k. With this, the partition function (2.25) can be written in terms of $2p'$ non-coincident integers as

$$Z_{p,q}\Big(A, \frac{r}{s}\Big) = \sum_{\substack{\boldsymbol{a}\in\mathbb{Z}_s^{p'} \\ a_b\neq a_c}} \mathrm{e}^{\,-\frac{2\pi\,\mathrm{i}}{s}\sum_b a_b} \int_0^1 \mathrm{d}\lambda\ \mathrm{e}^{\,-2\pi\,\mathrm{i}\,\lambda q} \sum_{\substack{\boldsymbol{m}\in\mathbb{Z}^{p'} \\ m_b>m_c}} \mathrm{e}^{\,-\frac{g^2 A}{2s}\sum_b (m_b - r a_b/s + \lambda)^2}\,. \tag{2.27}$$

The particularly interesting aspect of the expansion (2.27) is that, upon comparison with the heat kernel expansion (2.22), it shows that the Young tableaux row integers m_b are replaced in the noncommutative case by the fractional, $\frac{1}{s}$-valued variables $m_b + r\,a_b/s$. The momentum integers a_b also appear in this case to parameterize states of some $\mathbb{Z}_s$-valued theta-sector of the two-dimensional gauge theory, as would arise from an $\mathrm{SU}(s)$ structure group. This illustrates how noncommutativity deforms group representation quantities, similarly to the way in which it alters the dimensions of projective modules over the torus.

We can take this description of "noncommutative group theory" even further by exploiting Morita equivalence with commutative Yang-Mills theory in this case. The discrete Möbius transformation which maps the noncommutativity parameter $\theta = r/s$ to $\theta' = 0$ is parameterized by the $\mathrm{SL}(2,\mathbb{Z})$ matrix

$$\begin{pmatrix} s & -r \\ r' & s' \end{pmatrix} , \quad r\,r' + s\,s' = 1\,. \tag{2.28}$$

The topological numbers associated to any partition of a given Heisenberg module transform as $\mathrm{SL}(2,\mathbb{Z})$ doublets under the gauge extension of Morita equivalence,

$$\begin{pmatrix} \boldsymbol{p}' \\ \boldsymbol{q}' \end{pmatrix} = \begin{pmatrix} s & -r \\ r' & s' \end{pmatrix} \begin{pmatrix} \boldsymbol{p} \\ \boldsymbol{q} \end{pmatrix} , \tag{2.29}$$

so that by Poisson duality we also have

$$\begin{pmatrix} \boldsymbol{m}' \\ \boldsymbol{n}' \end{pmatrix} = \begin{pmatrix} s & -r \\ r' & s' \end{pmatrix} \begin{pmatrix} \boldsymbol{m} \\ \boldsymbol{n} \end{pmatrix} . \tag{2.30}$$

This gives a definition of the action of Morita duality on the algebraic numbers of the noncommutative gauge theory, rather than the conventional definition in terms of topological numbers. In addition, the dimension variables and the dimensionless Yang-Mills coupling constant transform in this particular case as

$$\begin{aligned} z_k' &= s\, z_k\,, \\ \left(g^2 A\right)' &= \frac{g^2 A}{s^3}\,. \end{aligned} \tag{2.31}$$

With these transformation rules, we can obtain a concise proof of the relation between the partition function (2.27) at rational θ and a commutative Yang-Mills theory. Using the transformation rules (2.30) of $\mathrm{SL}(2,\mathbb{Z})$ doublets of dual topological numbers (m_b, a_b), along with (2.31), we obtain from (2.27) the Morita dual partition function

$$Z_{p,q}\left(A, \frac{r}{s}\right) = \sum_{m_1 > \cdots > m_{p'}} \mathrm{e}^{\,\frac{2\pi\,\mathrm{i}\,r'}{s} \sum_b m_b'} \int_0^1 \mathrm{d}\lambda\, \mathrm{e}^{\,-2\pi\,\mathrm{i}\,\lambda\,q}\, \mathrm{e}^{\,-\frac{(g^2 A)'}{2} \sum_b (m_b' + s\lambda)^2}\,. \tag{2.32}$$

By rescaling $\lambda = \lambda'/s$, taking advantage of the periodicity of the integrand and using the transformation rule for the topological numbers (2.29), we may bring (2.32) into the form

$$Z_{p,q}\left(A, \frac{r}{s}\right) = \int_0^1 \mathrm{d}\lambda'\, \mathrm{e}^{\,-2\pi\,\mathrm{i}\,\lambda'\,q'} \sum_{m_1 > \cdots > m_{p'}} \mathrm{e}^{\,-\frac{(g^2 A)'}{2} \sum_b (m_b' + \lambda')^2}\, \mathrm{e}^{\,\frac{2\pi\,\mathrm{i}\,r'}{s} \sum_b (m_b' + \lambda')}\,. \tag{2.33}$$

Comparing with the commutative partition function (2.22), we find that (2.33) differs by an exponential factor which indicates the presence of a background flux. The effect of this flux can be made more explicit by absorbing it into quadratic terms in the energy and by

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again shifting λ'. The final result is

$$Z_{p,q}\left(A,\frac{r}{s}\right) = \mathrm{e}^{-\frac{2\pi^2}{(g^2A)'}\frac{r'}{s^2}(r'p'-2sq')}\int_0^1 \mathrm{d}\lambda'\,\mathrm{e}^{-2\pi\,\mathrm{i}\,\lambda' q'}\sum_{m_1>\cdots>m_{p'}}\mathrm{e}^{-\frac{(g^2A)'}{2}\sum_b(m_b'+\lambda')^2}\,. \tag{2.34}$$

This form differs from the commutative result (2.22) only by a global exponential prefactor which is cancelled by the dual of the background flux $b_{p,q}$ in (2.1) that has been dropped from the present formulas. In this way we have rederived the well-known duality between noncommutative Yang-Mills theory with coupling g^2A, topological charge q and rational-valued noncommutativity parameter $\theta = r/s$, and ordinary non-abelian gauge theory with coupling constant g^2A/s^3 and structure group $\mathrm{U}(sp-rq)$.

2.4 Higher casimir operators at irrational θ

For irrational values of the noncommutativity parameter θ, it is not possible to reduce the dual partition indices appearing in the quadratic form of (2.10) to a single integer, as the electric flux variables $m_k - n_k\theta$ cannot be decoupled in this case. The partition function is always intrinsically a sum over the two-dimensional dual lattice $\mathbb{Z}^2$ of K-theory charges. In the commutative and rational cases the K-theory group K_0 is isomorphic to $\mathbb{Z}\oplus\mathbb{Z}$. Hence the charges are independent and may be decoupled. In the irrational case the K-theory is instead naturally isomorphic to the group $\mathbb{Z}+\mathbb{Z}\theta\subset\mathbb{R}$, and thus the fluxes are topologically tied together. As we described in the previous subsection, we may consider the effect of noncommutativity as deforming Young tableaux row integers m_k to non-integral ones $m_k - n_k\theta$. As we will discuss further later on, this property is a direct group theoretical reflection of the known mathematical property that there is no well-defined notion of a structure group in noncommutative gauge theory, because of the mixing between spacetime and colour degrees of freedom through noncommutative gauge transformations. Indeed, this is the main reason for the lack of a definition of "physical" Yang-Mills theory on the noncommutative torus.

The group theoretic form (2.27) of the rational partition function illustrates this point in an interesting way. Let us consider the irrational number θ as the limit of large positive integers r and s with $\theta = \lim_{r,s} r/s$ fixed. The exponential sums in (2.27) over b essentially run from 0 to $s(p-q\theta)$ and are each weighted with a factor of $1/s$. In the limit $s\to\infty$, these sums turn into integrals and the partition function (2.27) can be written as a path integral

$$\begin{aligned} Z_{p,q}(A,\theta) = \prod_x \int_{\substack{m(x)\in\mathbb{Z}\\ m(x)\neq m(y)}}\mathrm{d}m(x)\int_{\substack{a(x)\in\mathbb{Z}_+\\ a(x)\neq a(y)}}\mathrm{d}a(x)\int_0^1\mathrm{d}\lambda\,\mathrm{e}^{-2\pi\,\mathrm{i}\,\lambda\,q}\times \\ \times\exp\left[-2\pi\,\mathrm{i}\int_0^{p-q\theta}\mathrm{d}x\,a(x) - \frac{g^2A}{2}\int_0^{p-q\theta}\mathrm{d}x\Big(m(x)-\theta a(x)+\lambda\Big)^2\right]. \end{aligned} \tag{2.35}$$

Note that the fields in (2.35) are integer-valued and obey an "exclusion" principle which ensures that there are no continuous fields in this mechanical system. This model defines a constrained but otherwise gaussian system, which resembles very much that which is obtained as the large-N limit of Young tableaux in the planar limit of the heat kernel representation of ordinary Yang-Mills theory [22]. Again this illustrates the deformation of the group theoretic representation by noncommutativity.

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To understand this latter point better, let us return to the intermediate form (2.8) of the resummed partition function. Formally performing the integrals over the dimension variables z_k yields the statistical sum

$$\begin{aligned} Z_{p,q}(A,\theta)' = &\sum_{\boldsymbol{p},\boldsymbol{m}}{}' \frac{(-1)^{|\boldsymbol{\nu}|}}{\prod_a \nu_a!}\delta_{p\,,\sum_k p_k}\int_0^1 \mathrm{d}\lambda\,\mathrm{e}^{\,-2\pi\,\mathrm{i}\,\lambda\,q}\times \\ &\times\prod_{k=1}^{|\boldsymbol{\nu}|}\frac{\sqrt{\pi}}{|p_k|}\exp\Big\{-\frac{2}{g^2A\theta^2}\Big[2\pi|p_k|\sqrt{\pi^2+\pi\,\mathrm{i}\,g^2A\theta(m_k+\lambda)}- \\ &\qquad -p_k\Big(2\pi^2+\pi\,\mathrm{i}\,g^2A\theta(m_k+\lambda)\Big)\Big]\Big\}. \end{aligned} \tag{2.36}$$

The result (2.36) omits from the sum those modules with vanishing p_k. The full result is not so straightforward to express in a simple way. However, this does not affect the expansion of (2.36) in the limit $g^2A\,\theta\to 0$, to which we now turn. In that limit, for positive p_k the exponential prefactor is identical to that of commutative gauge theory, while the contributions from partitions with $p_k\le 0$ are exponentially suppressed. To leading orders in $g^2A\,\theta$, we may then write the full partition function as

$$\begin{aligned} \lim_{g^2A\theta\to 0} Z_{p,q}(A,\theta) = &\sum_{\substack{\boldsymbol{p}\in\mathbb{Z}_+^p\\ \sum_k p_k=p}}\ \sum_{\boldsymbol{m}\in\mathbb{Z}^p}\frac{(-1)^{|\boldsymbol{\nu}|}}{\prod_a\nu_a!}\int_0^1\mathrm{d}\lambda\,\mathrm{e}^{\,-2\pi\,\mathrm{i}\,\lambda\,q}\prod_{k=1}^{|\boldsymbol{\nu}|}\frac{\sqrt{\pi}}{p_k}\times \\ &\times\exp\Big\{-\frac{g^2A}{2}p_k\Big[(m_k+\lambda)^2-\mathrm{i}\,\frac{g^2A\theta}{2\pi}(m_k+\lambda)^3-\frac{5(g^2A\theta)^2}{16\pi^2}(m_k+\lambda)^4+ \\ &\qquad +O\Big(\big(g^2A\theta\big)^3\Big)\Big]\Big\}+O\Big(\mathrm{e}^{\,-1/g^2A\theta^2}\Big). \end{aligned} \tag{2.37}$$

The expansion (2.37) gives a nice interpretation of the noncommutative gauge theory as a particular modification of ordinary Yang-Mills theory in the limit of small $g^2A\,\theta$. Notice first of all that the corrections to the commutative constraint $p_k>0$ are non-perturbative, exactly as is expected from the instanton representation. This gives a very clear picture of the dangers in viewing the theory as a perturbative expansion in the noncommutativity parameter θ. The m_k's to this order then have the usual interpretation as Young tableaux row integers for $\mathrm{U}(p)$, while the sums over the p_k's can be interpreted, as before, as a sum over conjugacy classes enforcing the non-coincidence of the m_k's. Rewriting (2.37) with this restriction and dropping irrelevant constants leads to a transparent form of the perturbative corrections to commutative Yang-Mills theory as

$$\begin{aligned} \lim_{g^2A\theta\to 0} Z_{p,q}(A,\theta) = &\int_0^1\mathrm{d}\lambda\,\mathrm{e}^{\,-2\pi\,\mathrm{i}\,\lambda q}\sum_{m_1>\cdots>m_p}\times \\ &\times\exp\Big\{-\frac{g^2A}{2}\sum_{a=1}^p\Big[(m_a+\lambda)^2-\sum_{\ell=1}^\infty\frac{8\,\mathrm{i}^{\,\ell}\left(1-\frac12\right)\left(2-\frac12\right)\cdots\left(\ell-\frac12\right)}{\pi^\ell(\ell+2)!}\times \\ &\qquad\times(g^2A\theta)^\ell(m_a+\lambda)^{\ell+2}\Big]\Big\}+O(\mathrm{e}^{\,-1/g^2A\theta^2}). \end{aligned} \tag{2.38}$$

The argument of the exponential in (2.38) can be interpreted in terms of Casimir operators. Generally, the $\ell^{\rm th}$ Casimir operator eigenvalue in the representation R of the unitary group $\mathrm{U}(p)$ characterized by highest weight components m_a is given by

$$C_\ell(R)=C_\ell(\boldsymbol{m})=\sum_{a=1}^{p}(m_a)^\ell\,\prod_{b\neq a}\left(1-\frac{1}{m_a-m_b}\right)\,. \tag{2.39}$$

Any polynomial in the Young tableaux row integers can be written as a linear combination of Casimir invariants (2.39) of the corresponding representation. The leading term of the exponential in (2.38) corresponds to the quadratic Casimir $C_2(\boldsymbol{m})$ and yields the expected commutative result (2.22). The higher-order corrections in $g^2A\,\theta$ involve higher Casimir invariants. Thus the partition function in the limit of small $g^2A\,\theta$ can be thought of as coming from the modification of the commutative Yang-Mills action by an infinite number of higher Casimir operators and the addition of other non-perturbative contributions. In commutative lattice gauge theory, higher Casimir invariants correspond to the presence of higher powers of the field strength F in the action [4]. The inclusion of higher Casimir operators in ordinary, two-dimensional Yang-Mills theory leads to generalized Yang-Mills theories [16]–[18] which are defined in the case of the torus by partition functions of the form

$$Z_p\Bigl(A\,,\,\{t_\ell\}\Bigr)=\sum_R\exp\left[-\frac{A}{2}\,\sum_{\ell>0}t_\ell\,C_\ell(R)\right]\,. \tag{2.40}$$

Therefore, the expansion (2.38) presents a resummation of the long-range effects of noncommutative gauge theory into local contributions governed by a generalized Yang-Mills theory (with infinitely-many Casimir operators), plus true non-local effects which appear non-perturbatively.

3. Polyakov loop correlators on the torus

We will now proceed to the explicit calculation of observables of the noncommutative gauge theory. For this, we shall exploit the fact that spacetime noncommutativity acts in a diagonal way in the instanton presentation of two-dimensional Yang-Mills theory [13], i.e. solutions of the Yang-Mills equations are modified in a way which preserves practically all structures present in the commutative theory. The deformation of the theory due to noncommutativity does not mix solutions of the equations of motion, nor the components (submodules) which comprise these solutions. This makes the instanton picture uniquely well-suited for understanding the noncommutative gauge theory. The key issue now is to understand how to use this feature to study Wilson loop correlators of the noncommutative theory.

While exact, group theoretic expressions are available in ordinary $\mathrm{U}(p)$ Yang-Mills theory on the torus for the correlators of any number of Wilson loop observables on arbitrary contours [23], not all of these are well-suited for an appropriate transcription to the noncommutative theory. For instance, topologically trivial Wilson loops lead to dimension factors in the sums over representations, which hinders a straightforward Poisson resummation to the instanton expansion. Furthermore, the fusion numbers for the decomposition of a direct product into irreducible representations are not known explicitly for arbitrary U(p) representations. Finally, one needs to build an expression which is covariant under gauge Morita duality transformations in order to be able to make the generalization to the noncommutative case. In this section we will compute a class of correlation functions in commutative U(p) gauge theory on $\mathbf{T}^2$ which admits an explicit expansion as a sum over instanton contributions and which admits a Morita covariant formulation.

These correlation functions will be standard Polyakov loop-anti-loop correlators interpreted in terms of instanton sums and restricted to a particular topological sector. Naturally, these correlators depend on areas separating the loops on the torus but, for the purposes of constructing Morita covariant objects, it will be necessary to consider the Fourier transform of this area dependence. The result of the calculations in this section will be an explicit formula for a Polyakov loop-anti-loop correlator written in terms of topological aspects (dimension and Chern number) of the gauge theory and "kinematic" data of the loops (winding number and discrete relative momentum). In the following section we will explicitly detail how this somewhat peculiar parameterization of a Polyakov loop correlator gives a direct route to open Wilson line correlators in noncommutative gauge theory via Morita equivalence.

3.1 Physical correlation functions

Consider the Polyakov loop operator [24, 25] of winding number $m \in \mathbb{Z}$,

$$\mathsf{P}_m(U) = \chi_{F^{\otimes m}}(U) = \mathrm{Tr}\, U^m \,, \tag{3.1}$$

which is defined as the character of the holonomy U of the gauge field around the inner cycle of the torus in the direct product $F^{\otimes m}$, where F is the fundamental representation of the U(p) structure group. From a kinematic point of view, the winding number m is the length of the single row of boxes in the Young diagram for the representation $F^{\otimes m}$. The set of all Polyakov loops (3.1) in ordinary Yang-Mills theory is thereby a complete set of characters for the (reducible) representations of the structure group. They form an algebraic basis which is equivalent to the complete set of characters of the $\ell^{\rm th}$ antisymmeterized fundamental representations $F^{\wedge\ell}$, owing to the relationships [26]

$$\chi_{F^{\wedge\ell}}(U) = \frac{(-1)^\ell}{\ell!}\,\frac{\partial^\ell}{\partial z^\ell}\exp\left(-\sum_{m=1}^{\infty}\frac{\mathsf{P}_m(U)}{m}\,z^m\right)\Bigg|_{z=0} . \tag{3.2}$$

All information required about the gauge group and observables can be written in terms of their correlators, and their winding numbers give a complete set of labels for representations of the commutative U(p) gauge group.

From a dynamical perspective, the Polyakov loop on a torus can be thought of as a physical charge sitting at some point on a spatial circle, with the integer-valued winding numbers giving the spectrum of colour-electric charges in the system. Because of the non-vanishing U(1) charge of a single loop, the expectation value of the loop operator (3.1) vanishes in the confining U(p) gauge theory, $\langle\mathsf{P}_m(U)\rangle = 0$, implying that it would require an infinite amount of energy to introduce a single fundamental test charge into the system. By charge conservation, the simplest non-trivial correlation function involving the operators (3.1) is the loop anti-loop correlator [27]

$$W_{p;m}(A_1,A_2)=\Big\langle \mathsf{P}_m(U)\,\mathsf{P}_{-m}(V)\Big\rangle\,. \tag{3.3}$$

This correlation function computes the interaction amplitude between a quark anti-quark pair in the $\mathrm{U}(p)$ gauge theory.

To evaluate (3.3), we note that the loops split the torus $\mathbf{T}^2$ into two cylinders of areas A_1 and A_2 such that

$$A=A_1+A_2 \tag{3.4}$$

is the total area of the torus. By using the cylinder amplitude (2.12) we can thereby write the normalized expectation value (3.3) as

$$W_{p;m}(A_1,A_2)=\frac{1}{Z_p(A)}\int_{\mathrm{U}(p)}[\mathrm{d}U][\mathrm{d}V]K_p(A_1;U,V)K_p(A_2;V,U)\mathsf{P}_m(U)\mathsf{P}_{-m}(V)\,. \tag{3.5}$$

The integrations over the unitary group in (3.5) can be evaluated by noting that products of characters may be written using multiplicativity and linearity as

$$\chi_R(U)\,\chi_S(U)=\chi_{R\otimes S}(U)=\sum_{R'}N_{R,S}^{R'}\,\chi_{R'}(U)\,, \tag{3.6}$$

where $N_{R,S}^{R'}$ are the fusion numbers which count the degeneracy of the irreducible representation R' in the Clebsch-Gordan decomposition $R\otimes S=\bigoplus_{R'}N_{R,S}^{R'}\,R'$. By applying the orthogonality relations (2.13) they may be defined as

$$N_{R,S}^{R'}=\int_{\mathrm{U}(p)}[\mathrm{d}U]\chi_R(U)\,\chi_S(U)\chi_{R'}^*(U)\,. \tag{3.7}$$

By substituting (2.12) and (3.1) into (3.5), and using (3.7), we arrive at the group theory presentation

$$W_{p;m}(A_1,A_2)=\frac{1}{Z_p(A)}\sum_{R,R'}N_{R'\,,\,F^{\otimes m}}^{R}\,N_{R\,,\,\overline{F}^{\otimes m}}^{R'}\ \mathrm{e}^{\,-\frac{g^2A_1}{2}\,C_2(R)-\frac{g^2A_2}{2}\,C_2(R')}\,. \tag{3.8}$$

The fusion numbers for the unitary group are sums of products of simple delta-functions in the Young tableaux row integers labelling each irreducible representation. Since the Casimir operators C_2 are quadratic in the row variables, the calculation of the loop correlators (3.8) involves only theta-functions on $\mathbf{T}^2$, to which Jacobi inversion may be applied.

To compute the fusion numbers in (3.8), we use the Weyl formula for the $\mathrm{U}(p)$ characters

$$\chi_R(U)=\chi_{\boldsymbol{m}}\left[\,\mathrm{e}^{\,2\pi\,\mathrm{i}\,\lambda}\right]=\frac{1}{\Delta\left[\,\mathrm{e}^{\,2\pi\,\mathrm{i}\,\lambda}\right]}\ \det_{1\le a,b\le p}\left(\,\mathrm{e}^{\,2\pi\,\mathrm{i}\,m_a\lambda_b}\right)\,, \tag{3.9}$$

where $\mathrm{e}^{\,2\pi\,\mathrm{i}\,\lambda_a}$, $\lambda_a \in [0,1]$, $a = 1,\dots,p$ are the eigenvalues of the unitary matrix U and

$$\Delta\left[\,\mathrm{e}^{\,2\pi\,\mathrm{i}\,\lambda}\right] = \prod_{a<b}\left(\mathrm{e}^{\,2\pi\,\mathrm{i}\,\lambda_a} - \mathrm{e}^{\,2\pi\,\mathrm{i}\,\lambda_b}\right) . \tag{3.10}$$

We then transform the unitary integration in (3.7) into an integration over the eigenvalues of U, with jacobian $\Delta[\,\mathrm{e}^{\,2\pi\,\mathrm{i}\,\lambda}]^2$, and substitute in (3.1) and (3.9) to write (3.8) as

$$\begin{aligned} W_{p;m}(A_1,A_2) \;=\; & \frac{1}{(p!)^2 Z_p(A)} \sum_{\boldsymbol{m},\boldsymbol{m}'\in\mathbb{Z}^p} \mathrm{e}^{\,-\frac{g^2 A_1}{2} C_2(\boldsymbol{m}) - \frac{g^2 A_2}{2} C_2(\boldsymbol{m}')} \times \\ & \times \prod_{a=1}^{p} \int_0^1 \mathrm{d}\lambda_a \mathrm{d}\mu_a \left(\sum_{c=1}^{p} \mathrm{e}^{\,2\pi\,\mathrm{i}\,m\lambda_c}\right)\left(\sum_{d=1}^{p} \mathrm{e}^{\,-2\pi\,\mathrm{i}\,m\mu_d}\right) \times \\ & \times \det_{1\le a,b\le p}\left(\mathrm{e}^{\,2\pi\,\mathrm{i}\,m_a\lambda_b}\right) \det_{1\le a,b\le p}\left(\mathrm{e}^{\,-2\pi\,\mathrm{i}\,m_a\mu_b}\right) \times \\ & \times \det_{1\le a,b\le p}\left(\mathrm{e}^{\,2\pi\,\mathrm{i}\,m'_a\mu_b}\right) \det_{1\le a,b\le p}\left(\mathrm{e}^{\,-2\pi\,\mathrm{i}\,m'_a\lambda_b}\right) . \end{aligned} \tag{3.11}$$

Here we have removed the ordering restriction (2.15) on the integers m_a and m'_a using the permutation symmetry of (2.16) and antisymmetry of the determinant factors. Note that the correlator (3.11) is symmetric under the simultaneous interchange of area labels $A_1 \leftrightarrow A_2$ and reflection of loop winding number $m \to -m$.

The product of determinants in (3.11) can be simplified by using the symmetry of the summand to combine them as

$$\begin{aligned} W_{p;m}(A_1,A_2) \;=\; & \frac{p!}{Z_p(A)} \sum_{\boldsymbol{m},\boldsymbol{m}'\in\mathbb{Z}^p} \mathrm{e}^{\,-\frac{g^2 A_1}{2} C_2(\boldsymbol{m}) - \frac{g^2 A_2}{2} C_2(\boldsymbol{m}')} \prod_{a=1}^{p}\int_0^1 \mathrm{d}\lambda_a \mathrm{d}\mu_a\, \mathrm{e}^{\,2\pi\,\mathrm{i}\,(m_a - m'_a)\lambda_a} \times \\ & \times \left(\sum_{c=1}^{p} \mathrm{e}^{\,2\pi\,\mathrm{i}\,m\lambda_c}\right)\left(\sum_{d=1}^{p} \mathrm{e}^{\,-2\pi\,\mathrm{i}\,m\mu_d}\right) \det_{1\le a,b\le p}\left(\mathrm{e}^{\,2\pi\,\mathrm{i}\,(m'_a - m_b)\mu_b}\right) . \end{aligned} \tag{3.12}$$

The integrations over λ_a in (3.12) can be used to fix the integers m'_a, for each summation index c, as $m'_a = m_a + m\,\delta_{a,c}$. The remaining integrals over μ_a then produce delta-functions in the m_a, and thereby reduce (3.11) to an expression involving only a single determinant as

$$\begin{aligned} W_{p;m}(A_1,A_2) \;=\; & \frac{p!}{Z_p(A)} \sum_{\boldsymbol{m}\in\mathbb{Z}^p} \sum_{c,d=1}^{p} \mathrm{e}^{\,-\frac{g^2 A}{2} C_2(\boldsymbol{m}) - \frac{g^2 A_1}{2}\left[m^2 - 2m\left(m_c - \frac{p-1}{2}\right)\right]} \times \\ & \times \det_{1\le a,b\le p}\left(\delta_{m_a,m_b + m\delta_{a,d} - m\delta_{a,c}}\right) . \end{aligned} \tag{3.13}$$

The expression (3.13) for the loop correlator involves a sum over $\boldsymbol{m} \in \mathbb{Z}^p$ of functions which are symmetric under permutation of the m_a's along with the determinant of a Kronecker delta-function. We recall that this feature was also common to the vacuum amplitude of section 2.2, which can be recovered from the sums in (3.13) at vanishing loop winding number $m = 0$.

Consequently, as with the partition function in (2.19), the correlation function (3.13) can be rewritten as a sum over conjugacy classes of the symmetric group S_p. With the notion of partition as explained in section 2.2, we can substitute the determinant expansion (2.17) into (3.13) to write it as a sum over partitions of the form

$$W_{p;m}(A_1,A_2) = \frac{p!}{Z_p(A)} \sum_{\boldsymbol{m}\in\mathbb{Z}^p} \sum_{\substack{\text{partitions}\\ \boldsymbol{p}}} \prod_{a=1}^{p} \frac{(-1)^{\nu_a}}{a^{\nu_a}\,\nu_a!} \prod_{k=1}^{|\boldsymbol{\nu}|} \mathrm{e}^{\,-\frac{g^2 A}{2} p_k \left(m_k - \frac{p-1}{2}\right)^2} \times \tag{3.14}$$
$$\times \sum_{l=1}^{|\boldsymbol{\nu}|} p_l \sum_{b=1}^{p_l} \mathrm{e}^{\,-\frac{g^2 A_1}{2} b\left[m^2 - 2m\left(m_l - \frac{p-1}{2}\right)\right] - \frac{g^2 A_2}{2}(b-1)\left[m^2 - 2m\left(m_l - \frac{p-1}{2}\right)\right]} \,.$$

Here and in the following we assume that $m \neq 0$. We have expressed the sums over the indices c and d in (3.13) as a sum over components b of distinct cycles in a partition. As before, the factors of p_k arise from the grouping of terms into cycles which subtracts off contributions whose row variables m_k coincide.

To extract physical information from the correlator (3.14), we set $m = 1$ and take the size of the torus to infinity while holding fixed one of the areas, say A_1 [27]. This damps out all flux in the area A_2 leaving only colour-electric flux in A_1, and the correlator gives the binding energy of a quark anti-quark pair interacting on the plane $\mathbb{R}^2$. In this limit, only the lowest (singlet) representation survives the sums in (3.13), and one finds

$$\lim_{A_2\to\infty} W_{p;1}(A_1,A_2) = p^2\,\mathrm{e}^{\,-g^2 A_1 p (p^2+11)/12} \,. \tag{3.15}$$

Restoring a contribution to the quadratic Casimir which we have neglected, we can identify a linear confining potential between the fundamental representation quark anti-quark pair with familiar string tension $g^2 p$.

3.2 Instanton contributions

To rewrite (3.14) as a sum over instantons, we carry out Poisson resummations of the integers m_k using (2.7). The result for the sum over the dual integers q_k, restricted to principal $\mathrm{U}(p)$ bundles over $\mathbf{T}^2$ of fixed total Chern number $q = \sum_k q_k$, allows one to define the correlation functions of Polyakov loop operators in Yang-Mills theory on a projective module of fixed topological numbers (p,q). By incorporating the constraint on $\sum_k q_k$ in the definition of partition in (2.4) for $\theta = 0$, we can thereby write

$$W_{p,q;m}(A_1,A_2) = \frac{p!}{Z_{p,q}(A,0)} \sum_{\substack{\text{partitions}\\ (\boldsymbol{p},\boldsymbol{q})}} \frac{(-1)^{|\boldsymbol{\nu}|}}{\prod_a \nu_a!} \prod_{k=1}^{|\boldsymbol{\nu}|} \sqrt{\frac{2\pi^2}{g^2 A (p_k)^3}}\, \mathrm{e}^{\,-\frac{2\pi^2}{g^2 A}\frac{(q_k)^2}{p_k}} \times \tag{3.16}$$
$$\times \sum_{l=1}^{|\boldsymbol{\nu}|} p_l \sum_{b=1}^{p_l} \mathrm{e}^{\,\frac{1}{2Ap_l}\left(bA_1 + (b-1)A_2\right)\left[g^2 m^2\left(bA_1 + (b-1)A_2 - p_l A\right) - 4\pi\,\mathrm{i}\, m q_l\right]} \,.$$

The sum over partition components in (3.16) gives the classical contribution to the Polyakov loop, so that (3.16) can be interpreted as the statistical average over all instantons of the Polyakov loop. Note that for the $\mathrm{U}(p)$ gauge theory on the torus defined in a sector of fixed, non-trivial 't Hooft flux q, Polyakov loop operators (3.1) should be accompanied by the transition functions $(\Gamma_q)^m$,

$$\mathsf{P}_{q;m}(U) = \mathrm{Tr}\, U^m (\Gamma_q)^m \,. \tag{3.17}$$

Here Γ_q is the corresponding $\mathrm{SU}(p)$ twist-eating matrix along the inner cycle of $\mathbf{T}^2$ and its inclusion is necessary in order to ensure invariance of the correlation functions under large gauge transformations [3, 28, 29].

In order to make contact with the noncommutative theory later on, we will require objects which transform covariantly under Morita equivalence. Such an object can be constructed from the correlator (3.16) if we carry out a Fourier transform in the direction transverse to the loop windings. For this, on a torus of fixed total area (3.4), we set $A_1 = (1-x)A$, $A_2 = xA$, and integrate over $x \in [0,1]$. The result of this transform yields the pair correlator of Polyakov loops with winding number m and momentum $n \in \mathbb{Z}$ as

$$W_{p,q;m,n}(A,0) = \int_0^1 \mathrm{d}x \ \mathrm{e}^{\,-2\pi\,\mathrm{i}\,nx} W_{p,q;m}((1-x)A, xA)\,. \tag{3.18}$$

Upon substitution of (3.16) into (3.18), the integral over x can be evaluated in terms of error functions with the result

$$\begin{aligned} W_{p,q;m,n}(A,0) \;=\; & \frac{p!}{Z_{p,q}(A,0)} \sum_{\substack{\text{partitions}\\ (\boldsymbol{p},\boldsymbol{q})}} \frac{(-1)^{|\boldsymbol{\nu}|}}{\prod_a \nu_a!} \prod_{k=1}^{|\boldsymbol{\nu}|} \sqrt{\frac{2\pi^2}{g^2A(p_k)^3}}\ \mathrm{e}^{\,-\frac{2\pi^2}{g^2A}\frac{(q_k)^2}{p_k}} \times \\ & \times \sum_{l=1}^{|\boldsymbol{\nu}|} (p_l)^2 \sqrt{\frac{2\pi^2}{g^2Am^2p_l}} \exp\left[\frac{\Big(4\pi(mq_l - np_l) - \mathrm{i}\,g^2Am^2p_l\Big)^2}{8g^2Am^2p_l}\right] \times \\ & \times \sum_{b=1}^{p_l} \mathrm{i} \left(\mathrm{erf}\left[\frac{4\pi(mq_l - np_l) - \mathrm{i}\,g^2Am^2(2-2b+p_l)}{\sqrt{8g^2Am^2p_l}}\right] - \right. \\ & \qquad \left. - \,\mathrm{erf}\left[\frac{4\pi(mq_l - np_l) - \mathrm{i}\,g^2Am^2(p_l-2b)}{\sqrt{8g^2Am^2p_l}}\right]\right). \end{aligned} \tag{3.19}$$

Note that the sum in (3.19) over components b of each cycle in a given partition is telescopic and reduces to the initial and final terms in the series.

By discarding irrelevant constants, the final expression for the Polyakov loop correlator thereby reads

$$\begin{aligned} W_{p,q;m,n}(A,0) \;=\; & \frac{1}{Z_{p,q}(A,0)} \sum_{\substack{\text{partitions}\\ (\boldsymbol{p},\boldsymbol{q})}} \frac{(-1)^{|\boldsymbol{\nu}|}}{\prod_a \nu_a!} \prod_{k=1}^{|\boldsymbol{\nu}|} \sqrt{\frac{2\pi^2}{g^2A(p_k)^3}}\ \mathrm{e}^{\,-\frac{2\pi^2}{g^2A}\frac{(q_k)^2}{p_k}} \times \\ & \times \sum_{l=1}^{|\boldsymbol{\nu}|} (p_l)^2 \sqrt{\frac{2\pi^2}{g^2Am^2p_l}} \exp\left[\frac{\Big(4\pi\binom{m}{n}\wedge\binom{p_l}{q_l} - \mathrm{i}\,g^2Am^2p_l\Big)^2}{8g^2Am^2p_l}\right] \times \\ & \times \mathrm{Im}\left(\mathrm{erf}\left[\frac{4\pi\binom{m}{n}\wedge\binom{p_l}{q_l} + \mathrm{i}\,g^2A\,m^2p_l}{\sqrt{8g^2Am^2p_l}}\right]\right), \end{aligned} \tag{3.20}$$

where we have defined the cross-product of two-dimensional integer vectors by the determinant

$$\begin{pmatrix} a \\ b \end{pmatrix} \wedge \begin{pmatrix} a' \\ b' \end{pmatrix} = \det \begin{pmatrix} a & a' \\ b & b' \end{pmatrix}. \tag{3.21}$$

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The symmetry of the correlator under interchange of areas $A_1 \leftrightarrow A_2$ and $m \to -m$ is manifest in the momentum form (3.20), whereby the exchange of area labels is replaced by a reflection of momentum $n \to -n$. Note that, like the restriction to sectors of fixed magnetic flux q, the transverse Fourier transform of the loop correlator is superfluous here, because the physical correlation functions can always be recovered as the Fourier series

$$W_{p;m}(A - A_2, A_2) = \sum_{q=-\infty}^{\infty} \mathrm{e}^{\,\pi\,\mathrm{i}\,(p-1)q} \sum_{n=-\infty}^{\infty} \mathrm{e}^{\,2\pi\,\mathrm{i}\,n A_2/A} W_{p,q;m,n}(A, 0)\,. \tag{3.22}$$

In other words, the translation group of the torus, with U(1) characters labelled by the transverse momenta n, is completely decoupled from the structure group, with characters labelled by the winding numbers m. In this case the translation and rotation groups appear together only in a free, direct product decomposition.

The precise meaning of the correlation functions (3.20) is most naturally understood in the noncommutative gauge theory. In that case, the separation of colour and translational degrees of freedom is not possible, reflecting the inherent non-locality of the quantum field theory. Similarly to the case of the vacuum energy, the pair correlator in the noncommutative case only admits an unambiguous definition on a fixed projective module and for fixed momentum n. The physical meaning of this restriction will be elucidated in the next section.

4. Open Wilson line correlators

In this section we shall transcribe the results obtained in the previous subsection into the noncommutative gauge theory by adapting the technique of [13] to correlation functions. While the physical meaning of Polyakov loop correlators was discussed in section 3.1, their noncommutative cousins are a bit more intricate to describe. We will therefore begin with a description of what sort of physics will be described by the noncommutative correlation functions, and then present the exact analytic expressions for the pair correlators in this case.

4.1 Kinematics of noncommutative dipoles

The simplest way to understand the necessity for the transverse momentum label of correlators is to look at the Dirac quantization condition for electric flux in the noncommutative field theory [30, 31]. Let us briefly recall how this works, in a manner tailored to what we shall need in the following. The winding number $m \in \mathbb{Z}$ of a Polyakov loop (3.1) corresponds to the large gauge transformation $A \mapsto A + 2\pi\, m$ around the inner cycle of the torus $\mathbf{T}^2$. These transformations are generated in field space by the electric field operator $E = \delta/\delta A$ and can be realized in the commutative case by the U(1) gauge transformation

$$\Omega(y) = \mathrm{e}^{\,2\pi\,\mathrm{i}\,y}\,, \tag{4.1}$$

with $y \in [0, 1]$ the coordinate along the inner cycle.

However, in the noncommutative case the corresponding star-gauge transformation generated by (4.1) also acts on the non-zero modes of the gauge fields and would generate a translation in the spatial dependence $x \mapsto x - \theta\, m$ of the gauge field, where the coordinate $x \in [0,1]$ runs transverse to y. Requiring that states in the physical Hilbert space of the noncommutative field theory be gauge invariant thereby requires also a simultaneous overall translation in space, which may be implemented by the transverse momentum operator P. This is guaranteed if the zero modes of the operator $E + \theta\, P$, rather than the electric field itself, are integral. Therefore, the modified Dirac quantization condition on the electric field is

$$\int_0^1 \mathrm{d}x \mathrm{d}y \ \mathrm{Tr}\, E = m - n\,\theta \equiv e_{m,n} \,, \tag{4.2}$$

where $m \in \mathbb{Z}$ and

$$n = \int_0^1 \mathrm{d}x \mathrm{d}y \ \mathrm{Tr}\, P \in \mathbb{Z} \tag{4.3}$$

is the zero-mode of the transverse momentum operator. Note that since P generates geometrical translations of the torus, its zero-mode spectrum is still integral. The change (4.2) in the spectrum of colour-electric charges m is completely analogous to the way in which the commutative ranks p are modified to non-integer module dimensions $p - q\theta$ in the noncommutative case. It shows explicitly how Young tableaux row integers are modified to non-integer ones determined by the transverse translational zero-modes. Note that the gauge covariant momentum can be represented by the operator $F\,E$, whose zero mode spectrum is given by $n(p - q\theta)$.

The modified spectrum (4.2) of electric charges is due to the fact that the elementary quanta of the noncommutative gauge theory are described by weakly-interacting, non-local electric dipoles [14, 15], whose dipole moments $e_{m,n}$ are related to their center-of-mass momenta n through the relation (4.2). The dipole's motion is transverse to its extension as a rigid rod along the y-direction of $\mathbf{T}^2$. When $\theta = 0$ they become ordinary point-like quanta, with a complete decoupling of momentum and winding number. Similarly to the way that the Polyakov loops (3.1) correspond to fundamental charges on the torus, the dipole excitations are created and annihilated by the open Wilson line operators [32][3]

$$\mathsf{O}_{m,n}[U] = \int_0^1 \mathrm{d}x \mathrm{d}y \ \mathrm{Tr}\, U_\star(x,y;x,y+m-n\theta) \star \mathrm{e}^{-2\pi\,\mathrm{i}\,n x} \,, \tag{4.4}$$

where $U_\star(x,y;x,y+e_{m,n})$ is the noncommutative parallel transport operator defined along the straight open contour starting at the point $(x,y) \in \mathbf{T}^2$, winding m times around the y-direction, and then ending up at the shift $-n\theta$ from y. They implicitly include the path-ordered star-exponentials of the appropriate background abelian gauge field required to absorb the large gauge transformation which is present due to the non-vanishing magnetic charge q. The operators (4.4) are gauge-invariant but non-local. They also illustrate, via the requirement of star-gauge invariance, the necessity of the transverse Fourier transform in the noncommutative setting.

[3]Other aspects of open Wilson line operators may be found in [28, 33, 34, 19].

The energy of an electric dipole of moment (4.2) in the gauge background characterized by the topological numbers $(p,q) \in \mathbb{Z}^2$ can be obtained from the perturbative spectrum of the quantum hamiltonian on $\mathbb{R} \times \mathbf{T}^2$. At leading order in perturbation theory, one finds the ground state energy [35]

$$\mathcal{E}_{p,q;m,n}(A,\theta) = \frac{1}{p-q\theta}\left[\frac{g^2A}{2}\Big(m-n\theta\Big)^2 + \left|\binom{m}{n}\wedge\binom{p}{q}\right|\right]. \tag{4.5}$$

With our choice of subtraction of background flux in (2.1), the instanton contribution to this energy from the constant curvature of the Heisenberg module itself vanishes as in (2.3), because this sets the energy of the stable vacuum state (without dipole excitations) to zero, $\mathcal{E}_{p,q;0,0}(A,\theta) = 0$. The first term in (4.5) is the energy associated to zero modes and is simply the kinetic energy corresponding to the dipole moment $e_{m,n}$. The second term is the contribution from oscillatory modes and it gives the energy of the massless excitations of the dipole. It coincides with the zero mode spectrum of the operator $\mathrm{Tr}\, FE - \mathrm{Tr}\, F \,\mathrm{Tr}\, E$ on $\mathbf{T}^2$.

In the case of supersymmetric Yang-Mills theory on the noncommutative torus, the energy formula (4.5) can be computed directly from the central charges of the corresponding BPS algebra [31]. It is an exact result in this case and corresponds generically to a $\frac{1}{4}$-BPS state. These states also naturally arise in string theory, wherein the integer pair (p,q) represents (D0,D2) brane charges on $\mathbf{T}^2$ in gauge backgrounds, while (m,n) corresponds to the winding numbers and momenta of fundamental strings wrapping around the cycles of the torus. For $\frac{1}{4}$-BPS states, the string charges K must decompose into the particle charges as $K = mq - np = \binom{m}{n}\wedge\binom{p}{q}$ [36].

4.2 Exact dipole interaction amplitudes

We will now turn to the dynamics of electric dipoles in the noncommutative gauge theory. The interaction energy between a pair of dipoles can be computed by using the open Wilson lines (4.4), along with the conservation of momentum and electric charge in the gauge theory to define

$$\Big\langle \mathsf{O}_{m,n}[U]\, \mathsf{O}_{m',n'}[V]\Big\rangle \equiv \delta_{m,m'}\,\delta_{n,n'}\; W_{p,q;m,n}(A,\theta)\,. \tag{4.6}$$

When $\theta = r/s$ is a rational number, Morita duality provides a one-to-one correspondence between the Polyakov loop operators (3.17) in commutative Yang-Mills theory and the noncommutative open Wilson line operators (4.4) [3, 28, 29]. We can use this covariance to explicitly deduce the noncommutative version (4.6) of the amplitude (3.20). Under the gauge Morita equivalence transformations parameterized by the $\mathrm{SL}(2,\mathbb{Z})$ matrices (2.28), the pairs of integers (p_k,q_k) transform as $\mathrm{SL}(2,\mathbb{Z})$ doublets (2.29), while the dimensionless Yang-Mills coupling constant g^2A transforms according to (2.31). Looking at (3.20), we see that the combinations of variables $g^2A\,m^2p_k$ and $\binom{m}{n}\wedge\binom{p_k}{q_k}$ play a special role. Requiring these two sets of combinations to be Morita invariant is tantamount to demanding that the integer m transform like the dimension of a module and that the cross-product $\binom{m}{n}\wedge\binom{p_k}{q_k}$

be invariant. These requirements are met if the pair of integers (m,n) has the expected vector transformation rule under $\mathrm{SL}(2,\mathbb{Z})$ as in (2.30),

$$\begin{pmatrix} m' \\ n' \end{pmatrix} = \begin{pmatrix} s & -r \\ r' & s' \end{pmatrix} \begin{pmatrix} m \\ n \end{pmatrix} , \tag{4.7}$$

since the cross-product (3.21) is clearly an $\mathrm{SL}(2,\mathbb{Z})$ invariant.

With the identification of the transformation properties of (3.20) under gauge Morita equivalence, we can now apply the procedure that was used in [13] to determine the partition function of two-dimensional Yang-Mills theory on a noncommutative torus. The Morita transformations (2.29) and (4.7) immediately provide the result for all tori with rational noncommutativity parameters $\theta = r/s$ through the replacement of dimensions p_k and electric flux m with their noncommutative generalizations $p_k - q_k\,\theta$ and $m - n\,\theta$. The definition of a partition $(\boldsymbol{p},\boldsymbol{q})$ is modified to its full noncommutative form (2.4), and the components of a partition are identified with the topological numbers of projective modules. The Morita transform of the pair correlator (3.20) thereby gives

$$\begin{aligned} W_{p,q;m,n}(A,\theta) \, = \, & \frac{1}{Z_{p,q}(A,\theta)} \sum_{\substack{\text{partitions} \\ (\boldsymbol{p},\boldsymbol{q})}} \frac{(-1)^{|\boldsymbol{\nu}|}}{\prod_a \nu_a!} \prod_{k=1}^{|\boldsymbol{\nu}|} \sqrt{\frac{2\pi^2}{g^2 A (p_k - q_k\theta)^3}} \times \\ & \times \exp\left[-\frac{2\pi^2}{g^2 A} \sum_{k=1}^{|\boldsymbol{\nu}|} (p_k - q_k\theta) \left(\frac{q_k}{p_k - q_k\theta} - \frac{q}{p - q\theta} \right)^2 \right] \times \\ & \times \sum_{l=1}^{|\boldsymbol{\nu}|} (p_l - q_l\theta)^2 \sqrt{\frac{2\pi^2}{g^2 A (m - n\theta)^2 \, (p_l - q_l\theta)}} \times \\ & \times \exp\left[\frac{\left(4\pi \binom{m}{n} \wedge \binom{p_l}{q_l} - \mathrm{i}\, g^2 A (m - n\theta)^2 (p_l - q_l\theta) \right)^2}{8 g^2 A (m - n\theta)^2 (p_l - q_l\theta)} \right] \times \\ & \times \mathrm{Im} \left(\mathrm{erf} \left[\frac{4\pi \binom{m}{n} \wedge \binom{p_l}{q_l} + \mathrm{i}\, g^2 A (m - n\theta)^2 \, (p_l - q_l\theta)}{\sqrt{8 g^2 A (m - n\theta)^2 \, (p_l - q_l\theta)}} \right] \right) , \end{aligned} \tag{4.8}$$

where we have taken into account the shift of the instanton contributions to the action in (3.20) by the constant curvature (2.2) associated with the topological numbers $(p,q) \in \mathbb{Z}^2$, which is produced by the Morita mapping. With the vector $\mathrm{SL}(2,\mathbb{Z})$ transformation properties of $(\boldsymbol{p},\boldsymbol{q})$ and (m,n), the correlator (4.8) transforms as the square of a module dimension $(p - q\,\theta)^2$, as it should since it involves a double trace.

As was noted in [13] for the case of the vacuum energy, the expression (4.8) for the correlation function at rational values of θ can be naturally generalized to irrational θ. Since the correlation function can be expressed as a sum over contributions from solutions of the classical equations of motion parameterized by partitions $(\boldsymbol{p},\boldsymbol{q})$, the arguments given in [13] for the continuity in θ of the localization of the path integral onto classical gauge field configurations apply here as well. These arguments strongly suggest that (4.8) defines the pair correlation functions of open Wilson line operators for *all* values of θ. In the following we will assume this to be the case. With this we require that $m - n\,\theta \neq 0$. In what follows it will prove more convenient to write the general correlator (4.8) in terms of an integral representation for the error function as

$$\begin{aligned} W_{p,q;m,n}(A,\theta) \,=\, & \frac{1}{Z_{p,q}(A,\theta)} \sum_{\substack{\text{partitions}\\ (\boldsymbol{p},\boldsymbol{q})}} \frac{(-1)^{|\boldsymbol{\nu}|}}{\prod_a \nu_a!} \prod_{k=1}^{|\boldsymbol{\nu}|} \sqrt{\frac{2\pi^2}{g^2 A (p_k - q_k\theta)^3}} \times \\ & \times \exp\left[-\frac{2\pi^2}{g^2 A} \sum_{k=1}^{|\boldsymbol{\nu}|} (p_k - q_k\theta) \left(\frac{q_k}{p_k - q_k\theta} - \frac{q}{p - q\theta}\right)^2\right] \times \\ & \times \sum_{l=1}^{|\boldsymbol{\nu}|} \frac{(p_l - q_l\theta)^2}{2} \int_{-1}^{1} \mathrm{d}s\, \mathrm{e}^{\,\pi\,\mathrm{i}\,(1-s)\binom{m}{n}\wedge\binom{p_l}{q_l} - g^2 A (m - n\,\theta)^2 (p_l - q_l\theta)(1-s^2)/8} \,. \end{aligned} \tag{4.9}$$

4.3 Dipole vacua

Armed with the exact analytic expressions for the open Wilson lines, in the subsequent sections we shall describe some of their physical properties. The pair correlators (4.8) and (4.9) naturally involve, for each component (p_k, q_k) of each partition $(\boldsymbol{p}, \boldsymbol{q})$, both the associated dipole kinetic energy and massless oscillations that appear in the energy formula (4.5). As we will see, the latter contributions play a prominent role in the instanton expansion, and so we will first briefly discuss in this subsection some further aspects of them. What will be particularly important in the following is that in the dipole vacuum state whereby the oscillatory modes vanish,

$$\binom{m}{n} \wedge \binom{p}{q} = 0\,, \tag{4.10}$$

the dipole kinetic energy corresponds to a maximally supersymmetric $\frac{1}{2}$-BPS state [31], just like the instanton configuration $(\boldsymbol{p}, \boldsymbol{q}) = (p, q)$ corresponding to the constant curvature gauge fields of the projective module. From (4.10) it follows that the transverse dipole momentum of this state is given by $n = n_0$, where

$$n_0 = e_{m,n_0} b_{p,q}\,. \tag{4.11}$$

The right-hand side of (4.11) is just the Poynting vector of the gauge background with constant magnetic flux (2.2) and noncommutative electric flux (4.2). In other words, the dipole vacuum is a maximally supersymmetric state characterized by the condition that its transverse momentum cancels exactly the momentum of the background electromagnetic field in the gauge vacuum.

As the cross-products in (4.8) and (4.9) produce gauge Morita invariants, it is natural to identify the product as a $\mathbb{Z}$-bilinear pairing between the abelian group of topological numbers (p_k, q_k) and the dual group consisting of geometrical loop winding and momentum numbers (m, n). In other words, we regard the product as a $(1,1)$ form on K-theory

$$\wedge\,:\, \mathrm{K}_0^* \times \mathrm{K}_0 \;\longrightarrow\; \mathbb{Z}\,, \tag{4.12}$$

where K_0^* is the Fourier dual of the K_0 lattice obtained via Poisson resummation as in section 2. Later on we shall see more evidence to support this claim. It is tempting to

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identify $\mathrm{K}_0^* = \mathrm{K}^0$ and (4.12) as the natural pairing between K-homology and K-theory. This would identify the pairs of integers (m,n) as the topological numbers corresponding to homotopy classes of Fredholm operators and the cross-product as the index map. The vanishing condition (4.10) would then imply the matching of chiral and anti-chiral zero modes of an associated Fredholm operator, as we expect for a state of maximal supersymmetry. Since this description is valid for any θ, it explicitly demonstrates, through Morita duality, why the cross-products play a role in the commutative correlation functions (3.20), even though their momentum dependence is superfluous. However, within the context of two-dimensional noncommutative gauge theory, the dipole picture will prove to be the more descriptive one. Indeed, as we will see, the pairs (m,n) are most naturally understood within a group theoretic setting analogous to those of section 2, in which the cross-products won't explicitly appear.

5. Moduli dependence of dipole correlation functions

In this section we will analyse the behaviours of the correlators obtained in the previous section in various limits of the coupling parameters of the noncommutative gauge theory, and thereby extracting their physical significance. In particular, we will develop the deformed group theory representation of the correlators and use this to justify the interpretations we have made previously, most notably in section 2. The connections with the commutative calculations of Polyakov loop correlators will thereby illuminate the group theoretic interpretation of two-dimensional noncommutative Yang-Mills theory.

5.1 Weak-coupling limit

The weak-coupling limit $g^2A \to 0$ of two-dimensional Yang-Mills theory defines a topological field theory [37]. This limit provides a method to not only calculate correlation functions in the topological field theory, but also to identify fundamental structures in the physical gauge theory. For example, in [13] the weak-coupling limit of the partition function was used to identify the moduli space of instantons for gauge theory on the noncommutative torus. In this subsection we will point out some properties of the weak-coupling limit of the open Wilson line correlator (4.8). This limit is most easily determined by using the integral representation (4.9).

As in the case of the vacuum amplitude (2.3), in the weak-coupling limit the sum over partitions in (4.9) is exponentially dominated by the partition containing only a single submodule, namely the projective module on which the noncommutative Yang-Mills theory is defined, $(\boldsymbol{p},\boldsymbol{q}) = (p,q)$. Consequently, the leading behaviour of the correlator (4.9) at weak-coupling is given by

$$\lim_{g^2A\to 0} W_{p,q;m,n}(A,\theta) = \frac{(p-q\theta)^2}{2}\int_{-1}^{1} \mathrm{d}s\,\mathrm{e}^{\,\pi\,\mathrm{i}\,(1-s)\binom{m}{n}\wedge\binom{p}{q}-g^2A(m-n\theta)^2(p-q\theta)(1-s^2)/8} + \\ +O\left(\mathrm{e}^{\,-1/g^2A}\right). \tag{5.1}$$

The integration in (5.1) depends crucially on whether or not the cross-product vanishes, as in (4.10), and so we must consider these two distinct cases separately.

First of all, a power series expansion in g^2A for non-vanishing $\binom{m}{n}\wedge\binom{p}{q}$ gives

$$\lim_{g^2A\to 0} W_{p,q;m,n}(A,\theta) = \frac{g^2A(m-n\theta)^2(p-q\theta)^3}{4\pi^2\left[\binom{m}{n}\wedge\binom{p}{q}\right]^2} + O\left(\left(g^2A\right)^2\right) + \\ + O\left(\mathrm{e}^{-1/g^2A}\right) \qquad \text{for } \binom{m}{n}\wedge\binom{p}{q}\neq 0\,, \tag{5.2}$$

and so in this case the correlator vanishes linearly with the dimensionless coupling constant. On the other hand, if (4.10) holds, then the correlator has a non-vanishing weak-coupling limit given by

$$\lim_{g^2A\to 0} W_{p,q;m,n}(A,\theta) = (p-q\theta)^2\left[1-\frac{g^2A}{12}(m-n\theta)^2(p-q\theta) + O\left(\left(g^2A\right)^2\right)\right] + \\ + O\left(\mathrm{e}^{-1/g^2A}\right) \qquad \text{for } \binom{m}{n}\wedge\binom{p}{q} = 0\,. \tag{5.3}$$

From these two cases we see the significance of the dipole vacuum state condition (4.10). The most intriguing aspect of these results is that they are essentially independent of the noncommutativity parameter θ. In particular, we see that even in the commutative theory there is a fundamental importance to the cross-product $\binom{m}{n}\wedge\binom{p}{q}$ which brings together loop winding number m and momentum n.

This analysis leads to a physical interpretation of the cross-product in the commutative theory, i.e. without recourse to dipole physics. The weak coupling-limit projects the quantum gauge theory onto a particular topological sector characterized by K-theory charges $(p,q)\in\mathbb{Z}^2$. This sector can support certain colour-electric charges and loop momenta $(m,n)\in\mathbb{Z}^2$. By "support" we mean that the projected theory can form a gauge-invariant singlet given the specific loop configuration. From (5.2) and (5.3) we see that $\binom{m}{n}\wedge\binom{p}{q}$ is exactly the parameter which determines if a singlet configuration is available for loops (m,n). If not, then the condition (4.10) is violated and the pair correlator vanishes. It is amusing to note that the weak-coupling limit of a loop configuration (m,n) is identical to that of $(k\,m\,,\,k\,n)$ for any integer $k\neq 0$. In this way the correlation function fixes a topological equivalence relation $(m,n)\sim(k\,m\,,\,k\,n)\ \ \forall k\in\mathbb{Z}-\{0\}$. This equivalence comes about from the fact that the singlet condition requires the integer vectors $\binom{m}{n}$ and $\binom{p}{q}$ to be parallel in the cross-product (3.21), so that the loops are commensurate with the bundle over the torus on which the gauge theory is defined.

In the physical commutative gauge theory, the group theory expansion (3.8) shows explicitly that the pair correlator always returns the result (5.3) at $\theta=0$ in the weak-coupling limit. This is because the physical gauge theory is defined as the double sum (3.22) over Chern number q and loop momentum n. There are always terms in these series with (p,q) and (m,n) satisfying the condition (4.10). Nevertheless, this property clearly demonstrates the importance of the loop momentum n, and the cross-product $\binom{m}{n}\wedge\binom{p}{q}$, in the restriction of the commutative gauge theory to a sector of fixed magnetic charge q. The theory defined on a fixed projective module (p,q), even in the commutative case, has a natural set of observables which are necessarily labelled by *two* integers (m,n), rather than just one.

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Going back to the dipole interpretation, we recall that the condition (4.10) turned the $\frac{1}{4}$-BPS dipole state into a $\frac{1}{2}$-BPS state. This is very natural given that the constant curvature condition defines $\frac{1}{2}$-BPS gauge field configurations. The weak-coupling limit in this way projects the full quantum field theory onto the Higgs branch of the moduli space of $\frac{1}{2}$-BPS solutions. The entire moduli space in the corresponding supersymmetric gauge theory may be viewed as a fibration over this branch. Maximal supersymmetry in this way controls which dipole configurations have non-trivial topological interactions. Alternatively, we may regard (4.10) as a requirement of momentum conservation. Only when the total momentum of the dipole and background electromagnetic field configuration is conserved do we obtain a non-vanishing scattering amplitude.

5.2 Strong-coupling expansion

The integral representation of the correlation function (4.9) allows us to carry out a resummation procedure analogous to that performed on the partition function in section 2.1. The result is a form of the correlation function which is amenable to investigations of the strong-coupling limit and physical interpretation of the correlators in noncommutative Yang-Mills theory. The calculation is identical to that of the partition function and so we shall only sketch the derivation.

We begin from (4.9) by imposing the constraints (2.4) defining partitions directly as in (2.5), and then again Fourier resolve the delta-functions over the topological charges q_k and module dimensions z_k as in (2.6) to get

$$
\begin{aligned}
W_{p,q;m,n}(A,\theta) =& \frac{1}{Z_{p,q}(A,\theta)} \sum_{\boldsymbol{p},\boldsymbol{q}} \frac{(-1)^{|\boldsymbol{\nu}|}}{\prod_a \nu_a!} \delta_{p,\sum_k p_k} \int_0^1 \mathrm{d}\lambda\, \mathrm{e}^{-2\pi\,\mathrm{i}\,\lambda\, q} \times \\
&\times \prod_{k=1}^{|\boldsymbol{\nu}|} \sqrt{\frac{2\pi^2}{g^2 A}} \int_{0^+}^{\infty} \frac{\mathrm{d}z_k}{(z_k)^{3/2}} \int_{-\infty}^{\infty} \frac{\mathrm{d}x_k}{2\pi}\, \mathrm{e}^{\,\mathrm{i}\,x_k(z_k - p_k + q_k\theta)}\, \mathrm{e}^{2\pi\,\mathrm{i}\,\lambda q_k - \frac{2\pi^2}{g^2 A}\frac{(q_k)^2}{z_k}} \times \\
&\times \sum_{l=1}^{|\boldsymbol{\nu}|} \frac{(z_l)^2}{2} \int_{-1}^{1} \mathrm{d}s\, \mathrm{e}^{\pi\,\mathrm{i}\,(1-s)(q_l(m-n\theta)-n z_l) - g^2 A z_k (m-n\theta)^2(1-s^2)/8}\,.
\end{aligned}
\tag{5.4}
$$

The q_k series can be Poisson resummed and the resulting gaussian integrations over the x_k variables carried out. The results are identical to what was found for the partition function in (2.8) except for a factor containing information about the Wilson lines. After a subsequent Poisson resummation of the p_k variables, we thereby arrive at

$$
\begin{aligned}
W_{p,q;m,n}(A,\theta) =& \frac{1}{Z_{p,q}(A,\theta)} \sum_{\boldsymbol{n},\boldsymbol{m}} \frac{(-1)^{|\boldsymbol{\nu}|}}{\prod_a \nu_a!} \int_0^1 \mathrm{d}\mu\, \mathrm{e}^{-2\pi\,\mathrm{i}\,\mu p} \int_0^1 \mathrm{d}\lambda\, \mathrm{e}^{-2\pi\,\mathrm{i}\,\lambda q} \times \\
&\times \prod_{k=1}^{|\boldsymbol{\nu}|} \int_{0^+}^{\infty} \frac{\mathrm{d}z_k}{z_k}\, \mathrm{e}^{-\frac{g^2 A}{2} z_k (m_k + \lambda + \theta(\mu - n_k))^2}\, \mathrm{e}^{2\pi\,\mathrm{i}\,z_k(\mu - n_k)} \times \\
&\times \sum_{l=1}^{|\boldsymbol{\nu}|} \frac{(z_l)^2}{2} \int_{-1}^{1} \mathrm{d}s\, \mathrm{e}^{-\pi\,\mathrm{i}\,(1-s) n z_l}\, \mathrm{e}^{-\frac{g^2 A}{4} z_l (1-s)[(m-n\theta)^2 + 2(m-n\theta)(m_l+\lambda+\theta(\mu-n_l))]}.
\end{aligned}
\tag{5.5}
$$

After a shift in λ and a change of variable in the s-integral we get

$$\begin{aligned} W_{p,q;m,n}(A,\theta) \;=\; & \frac{1}{Z_{p,q}(A,\theta)}\sum_{\boldsymbol{n},\boldsymbol{m}}\frac{(-1)^{|\boldsymbol{\nu}|}}{\prod_a \nu_a!}\int_0^1 \mathrm{d}\mu\,\mathrm{e}^{\,-2\pi\,\mathrm{i}\,\mu(p-q\theta)}\int_0^1 \mathrm{d}\lambda\,\mathrm{e}^{\,-2\pi\,\mathrm{i}\,\lambda q}\,\times \\ & \times\prod_{k=1}^{|\boldsymbol{\nu}|}\int_{0^+}^{\infty}\frac{\mathrm{d}z_k}{z_k}\,\mathrm{e}^{\,-\frac{g^2A}{2}z_k(m_k-n_k\theta+\lambda)^2}\,\mathrm{e}^{\,2\pi\,\mathrm{i}\,z_k(\mu-n_k)}\,\times \\ & \times\sum_{l=1}^{|\boldsymbol{\nu}|}(z_l)^2\int_0^1 \mathrm{d}t\,\mathrm{e}^{\,-2\pi\,\mathrm{i}\,tnz_l}\,\mathrm{e}^{\,-\frac{g^2A}{2}tz_l\left((m-n\theta+m_l-n_l\theta+\lambda)^2-(m_l-n_l\theta+\lambda)^2\right)}\,, \end{aligned} \tag{5.6}$$

and by explicitly performing the integral over t we arrive at the final result

$$\begin{aligned} W_{p,q;m,n}(A,\theta) \;=\; & \frac{1}{Z_{p,q}(A,\theta)}\sum_{\boldsymbol{n},\boldsymbol{m}}\frac{(-1)^{|\boldsymbol{\nu}|}}{\prod_a \nu_a!}\int_0^1 \mathrm{d}\mu\,\mathrm{e}^{\,-2\pi\,\mathrm{i}\,\mu(p-q\theta)}\int_0^1 \mathrm{d}\lambda\,\mathrm{e}^{\,-2\pi\,\mathrm{i}\,\lambda q}\,\times \\ & \times\prod_{k=1}^{|\boldsymbol{\nu}|}\int_{0^+}^{\infty}\frac{\mathrm{d}z_k}{z_k}\,\mathrm{e}^{\,-\frac{g^2A}{2}z_k(m_k-n_k\theta+\lambda)^2}\,\mathrm{e}^{\,2\pi\,\mathrm{i}\,z_k(\mu-n_k)}\,\times \\ & \times\sum_{l=1}^{|\boldsymbol{\nu}|}z_l\,\frac{1-\mathrm{e}^{\,-2\pi\,\mathrm{i}\,nz_l-\frac{g^2A}{2}z_l\left((m-n\theta)^2+2(m-n\theta)(m_l-n_l\theta+\lambda)\right)}}{2\pi\,\mathrm{i}\,n+\frac{g^2A}{2}\Big((m-n\theta)^2+2(m-n\theta)(m_l-n_l\theta+\lambda)\Big)}\,. \end{aligned} \tag{5.7}$$

From (5.6) it is clear that the noncommutative electric field variables $m-n\,\theta$ should be considered on the same footing as the dual topological numbers $m_k-n_k\,\theta$. This is consistent with what we have observed in section 4 when we transformed the correlation function in the commutative theory to one in the noncommutative theory. Not only do the pairs of integers (m,n) and (m_k,n_k) transform in the same manner under $\mathrm{SL}(2,\mathbb{Z})$ gauge Morita equivalence, but they should also both be thought of as algebraic numbers which are dual to topological numbers such as (p_k,q_k) that figure prominently in the definition of the correlation function (4.8). The fact that the analog of the cross-product (4.12) does not appear in (5.6) supports this identification.

In fact, the strong-coupling resummation (5.6) of the pair correlation function gives a striking example of how noncommutativity changes the definition of Yang-Mills theory on the torus. As discussed in section 3.1, in the commutative theory the winding number m is related to group theoretic quantities such as irreducible representations and the highest weights m_k which parameterize them. In the noncommutative case, these row lengths are generalized to two-dimensional objects $m_k-n_k\theta$ and in (5.6) there appears the analogous structure for the generalization of the winding number m, the noncommutative electric field $m-n\theta$. The definition of a Wilson line in noncommutative gauge theory thus does not depend solely on group theoretic quantities but also on dipole momenta. The correlation function (5.6) thereby gives an explicit example of how colour and spacetime degrees of freedom are non-trivially tied together in the noncommutative theory. The representation theory of the noncommutative gauge group is determined by including the group of translations on the torus with momentum integers into the usual colour representation labels, which can no longer be trivially separated. The twisting of the two group structures by noncommutativity into a single Morita covariant structure is strikingly similar to the realization of the noncommutative torus as the crossed product algebra of functions on the circle $\mathbf{S}^1$ by the integers $\mathbb{Z}$. The explicit mixing of the translation group with the gauge symmetries of the model is a manifestation of the teleparallelism property possessed by generic noncommutative gauge theories [38]. The arguments above also justify the claims made in section 2 that the dual topological integers (m_k, n_k) can be regarded as labelling the contributions from virtual electric dipoles. This is an exact realization of the proposal that the effective actions of generic noncommutative quantum field theories can be expressed in terms of open Wilson lines [39, 40, 41].

There are various other salient features that can be readily deduced from the strong-coupling form (5.6) of the open Wilson line correlator. For example, the selection and weighting of partitions in the instanton expansion (4.8) (or (4.9)) is identical to that of the partition function (2.3). Thus the arguments which established θ-smoothness of the vacuum amplitude [13] carry over to the pair correlators. This is completely evident in the strong-coupling resummation (5.6) (or (5.7)), which shows that the dipole amplitudes are analytic functions of the noncommutativity parameter. Thus the poles in θ which sometimes arise in perturbative expansions [5, 7, 42], and are the hallmark of UV/IR mixing in noncommutative field theories, appear to be non-perturbatively resummed into non-singular expressions in two-dimensional noncommutative Yang-Mills theory. This supports suggestions that the UV/IR mixing phenomenon is merely a resummable artifact of perturbation theory.

One other aspect which we will not cover in detail is the commutative limit of the open Wilson line correlator. This can be calculated from (5.4) using the techniques found in section 2.4 to calculate equation (2.36). The result is an expansion with exponentially small contributions in the limit of vanishing θ. This form will not explicitly depend on the areas between the loops but area dependence can be recovered by a Fourier transform to invert (3.18).

6. High-energy behaviour of open Wilson lines

Because the extent of a dipole grows with its momentum, at high energies the open Wilson lines become very long. Using this fact the correlators of large momentum open Wilson lines can be analysed perturbatively within a ladder approximation in the planar limit of the gauge theory. For example, in the case of Yang-Mills theory on noncommutative $\mathbb{R}^4$, the two-point functions of open Wilson line operators at weak coupling exhibit a universal exponential growth with high momentum as [19, 20]

$$\langle \mathsf{O}(k)\mathsf{O}(-k)\rangle_{4\mathrm{D}} \simeq \exp\left(\sqrt{\frac{\sqrt{g^2N}|k||\theta\cdot k|}{4\pi}} - \frac{g^2N|k\theta|\Lambda}{4\pi^2}\right). \tag{6.1}$$

This result implies, for fixed[4] cutoff Λ, an exponential suppression of higher correlation functions at large momentum, except for those whereby a pair of dipole momenta become

[4]Clearly, in order to remove all cutoff dependence one would need to consider ratios of higher-point

anti-parallel in which case the exponential growth is restored. The expression (6.1) is identical to that of the two-point correlation function computed using the supergravity dual of noncommutative $\mathcal{N}=4$ supersymmetric Yang-Mills theory, and it is reminiscent of the behaviour of high-energy fixed-angle scattering amplitudes in string theory.

The open Wilson lines can be analysed on noncommutative $\mathbb{R}^2$ in a particular decompactification limit of Yang-Mills theory on the noncommutative torus [9]. In the planar limit of the Morita dual theories which approximate the given noncommutative gauge theory through a sequence of rational approximants, i.e. $\theta = \lim_{r,s} r/s$, within a certain regime one can find an analogous exponential increase in the correlation functions with large momentum of the form

$$\langle \mathsf{O}(k)\mathsf{O}(-k)\rangle_{\rm 2D} \simeq \frac{\exp\left(\frac{g^2A}{2}|k|N\right)}{\frac{g^2A}{2}|k|N^2}\,, \qquad |k| < N-1\,. \tag{6.2}$$

This expression is valid at strong 't Hooft coupling, and is exact as a function of the dipole momentum $|k|$, receiving corrections only at order $1/N$. By redefining parameters it can be extrapolated to the weak coupling result $\exp(\sqrt{g^2ANk^2\theta^2/\pi})$ obtained by resumming all planar ladder diagrams in perturbation theory. Similar large momentum dependences have been observed in [8, 10]. In this final section we will use the exact expressions for the pair correlators obtained in this paper to study the high-energy behaviour of the dipole scattering amplitudes.

Within the framework of gauge theory on the noncommutative torus, it is in fact straightforward to prove that no such exponential rise with momentum can occur. This fact is based on the integral representation (4.9) from which it can be shown that the correlator is uniformly bounded as a function of the dimensionless Yang-Mills coupling constant g^2A. For this, we use the elementary integral inequality

$$\left|\frac{1}{2}\int_{-1}^{1}{\rm d}s\ {\rm e}^{\,\pi\,{\rm i}\,a(1-s)-b(1-s^2)}\right| \leq \frac{1}{2}\int_{-1}^{1}{\rm d}s\,{\rm e}^{\,-b(1-s^2)} \leq 1 \tag{6.3}$$

valid for any $a,b\in\mathbb{R}$, the definition (2.4) of partition, and the instanton expansion (2.3) of the vacuum energy to obtain the bound

$$\begin{aligned} W_{p,q;m,n}(A,\theta) \leq{}& \frac{1}{Z_{p,q}(A,\theta)}\sum_{\substack{\text{partitions}\\(\boldsymbol{p},\boldsymbol{q})}}\frac{(-1)^{|\boldsymbol{\nu}|}}{\prod_a \nu_a!}\prod_{k=1}^{|\boldsymbol{\nu}|}\sqrt{\frac{2\pi^2}{g^2A(p_k-q_k\theta)^3}}\times\\ &\times\exp\left[-\frac{2\pi^2}{g^2A}\sum_{k=1}^{|\boldsymbol{\nu}|}(p_k-q_k\theta)\left(\frac{q_k}{p_k-q_k\theta}-\frac{q}{p-q\theta}\right)^2\right]\times\\ &\times\sum_{l=1}^{|\boldsymbol{\nu}|}(p_l-q_l\theta)^2 \end{aligned}$$

correlation functions. Since the results for the two-dimensional theory in which we are primarily interested do not share such a UV cutoff ambiguity, we will restrict our comments to the four-dimensional case with a fixed cutoff.

$$
\begin{aligned}
&\leq \frac{(p-q\theta)^2}{Z_{p,q}(A,\theta)} \sum_{\substack{\text{partitions}\\ (\boldsymbol{p},\boldsymbol{q})}} \frac{(-1)^{|\boldsymbol{\nu}|}}{\prod_a \nu_a!} \prod_{k=1}^{|\boldsymbol{\nu}|} \sqrt{\frac{2\pi^2}{g^2A(p_k-q_k\theta)^3}} \times \\
&\quad \times \exp\left[-\frac{2\pi^2}{g^2A}\sum_{k=1}^{|\boldsymbol{\nu}|}(p_k-q_k\theta)\left(\frac{q_k}{p_k-q_k\theta}-\frac{q}{p-q\theta}\right)^2\right] \\
&= (p-q\theta)^2 \,.
\end{aligned}
\tag{6.4}
$$

The lack of a runaway behaviour in momentum is a reflection of the fact, established by the calculations of the present paper, that noncommutative Yang-Mills theory in two dimensions is a completely well-behaved (finite) quantum field theory.

In fact, we can go even further and show that the correlation function vanishes for high-energy dipoles. For this, we use the strong-coupling expansion (5.7) in the limit of large momentum n. For $\theta \neq 0$ we find

$$
\begin{aligned}
\lim_{n\to\infty} W_{p,q;m,n}(A,\theta) &= \frac{1}{Z_{p,q}(A,\theta)} \sum_{\boldsymbol{n},\boldsymbol{m}} \frac{(-1)^{|\boldsymbol{\nu}|}}{\prod_a \nu_a!} \int_0^1 \mathrm{d}\mu\, \mathrm{e}^{-2\pi\,\mathrm{i}\,\mu(p-q\theta)} \int_0^1 \mathrm{d}\lambda\, \mathrm{e}^{-2\pi\,\mathrm{i}\,\lambda q} \times \\
&\quad \times \prod_{k=1}^{|\boldsymbol{\nu}|} \int_{0^+}^{\infty} \frac{\mathrm{d}z_k}{z_k}\, \mathrm{e}^{-\frac{g^2A}{2} z_k(m_k-n_k\theta+\lambda)^2}\, \mathrm{e}^{2\pi\,\mathrm{i}\, z_k(\mu-n_k)} \times \\
&\quad \times \sum_{l=1}^{|\boldsymbol{\nu}|} \frac{2z_l}{g^2A(m-n\theta)^2} + O\Big((m-n\theta)^{-3}\Big) \\
&= \frac{2(p-q\theta)}{g^2A(m-n\theta)^2} + O\Big((m-n\theta)^{-3}\Big)\,.
\end{aligned}
\tag{6.5}
$$

The vanishing of the pair correlator in this case is more rapid than its commutative counterpart. For $\theta = 0$, the strong-coupling expansion (5.7) can be reduced to the form

$$
\begin{aligned}
W_{p,q;m,n}(A,0) &= \frac{1}{Z_{p,q}(A,0)} \sum_{\substack{\text{partitions}\\ \boldsymbol{p}}} \sum_{\boldsymbol{m}\in\mathbb{Z}^p} \frac{(-1)^{|\boldsymbol{\nu}|}}{\prod_a \nu_a!} \int_0^1 \mathrm{d}\lambda\, \mathrm{e}^{-2\pi\,\mathrm{i}\,\lambda q} \times \\
&\quad \times \prod_{k=1}^{|\boldsymbol{\nu}|} \frac{1}{p_k}\, \mathrm{e}^{-\frac{g^2A}{2} p_k(m_k+\lambda)^2} \sum_{l=1}^{|\boldsymbol{\nu}|} p_l \times \\
&\quad \times \sum_{a=1}^{p} \frac{1-\mathrm{e}^{-\frac{g^2A}{2} p_l\left(m^2+2m(m_a+\lambda)\right)}}{2\pi\,\mathrm{i}\, n + \frac{g^2A}{2}\left(m^2+2m(m_a+\lambda)\right)}\,,
\end{aligned}
\tag{6.6}
$$

which vanishes as $1/n$ for $n \to \infty$.

Even in the decompactification limit whereby the theory is projected out into gauge theory on the noncommutative plane, such momentum behaviour does not occur. This is because, like in the case of the partition function, the large area limit of (5.7) is trivial,

$$
\lim_{g^2A\to\infty} W_{p,q;m,n}(A,\theta) = 0\,. \tag{6.7}
$$

This result is most easily derived from the integral representation (4.9). Note that this vanishing of the momentum representation correlator at $\theta = 0$ is consistent with the limit of the position representation in (3.15) in the physical theory, due to the fact that the Fourier transforms (3.18) and (3.22) do not commute with taking the large area limit. Thus the momentum representation correlator can be trivial at large A without contradicting the non-triviality of the position representation correlator.

Consequently, the correlation functions of infinitely-long, anti-parallel open Wilson lines on the noncommutative plane vanish, and this is a reflection of the fact that the star-gauge invariance in this model contains area-preserving diffeomorphisms [13]. Thus the long (high momentum) dipole excitations in the present case play no role and are completely decoupled from the dynamics of the noncommutative gauge theory. This conclusion holds in the strong coupling regime of the gauge theory whereby the connections with string theory are expected to hold. However, this does not mean that the noncommutative field theory is free from stringy effects. For example, we have explicitly demonstrated in (2.10) and (5.7) that the observables of the system can expressed in terms of the non-local dipole quanta, with non-vanishing correlations at intermediate energy scales. Furthermore, we must also remember that the results (6.2) are derived in the planar $N \to \infty$ limit. The appropriate large-N limit of gauge theory on the noncommutative torus may well agree with these predictions.

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